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K_{l3} form factors at order p^6 of chiral perturbation theory

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Abstract. This paper describes the calculation of the semileptonic K_{l3} decay form factors at order p^6 of chiral perturbation theory, which is the next-to-leading order correction to the well-known p^4 result achieved by Gasser and Leutwyler. At order p^6 the chiral expansion contains one- and two-loop diagrams which are discussed in detail. The irreducible two-loop graphs of the sunset topology are calculated numerically. In addition, the chiral Lagrangian $\mathcal{L}^{(6)}$ produces direct couplings with the W bosons. Due to these unknown couplings, one can always add linear terms in q^2 to the predictions of the form factor $f_-(q^2)$. For the form factor $f_+(q^2)$, this ambiguity involves even quadratic terms. Making use of the fact that the pion electromagnetic form factor involves the same q^4 counterterm, the q^4 ambiguity can be resolved. Apart from the possibility of adding an arbitrary linear term in q^2 our calculation shows that chiral perturbation theory converges very well in this application, as the $\mathcal{O}(p^6)$ corrections are small. Comparing the predictions of chiral perturbation theory with the recent CPLEAR data, it is seen that the experimental form factor $f_+(q^2)$ is well described by a linear fit, but that the slope λ_+ is smaller by about 2 standard deviations than the $\mathcal{O}(p^4)$ prediction. The unavoidable q^2 counterterm of the $\mathcal{O}(p^6)$ corrections allows one to bring the predictions of chiral perturbation theory into perfect agreement with experiment.

1 Introduction

The hadronic matrix elements of weak decays constitute a decisive testing ground for our understanding of low-energy strong interactions. In this respect, the semileptonic K_{l3} decay is one of the cleanest and most interesting processes. In particular, it has been stressed [1] that this decay constitutes the best source for the extraction of the CKM matrix element $|V_{us}|$. On the experimental side there exist a number of old high statistics results (some contradictory), and new precision experiments are in progress or already published [2]. On the theoretical side chiral perturbation theory [3,4] has established itself as a powerful effective theory of low-energy strong interactions. Based on the symmetry of the underlying QCD, chiral perturbation theory produces a systematic low-energy expansion of the observables in this regime. Unfortunately, because of the non-renormalizability of the effective theory, higher powers in the energy expansion require higher loop Feynman integrals and as input an ever increasing number of renormalization constants. The p^4 Lagrangian involves ten free parameters which were determined in the fundamental papers of Gasser and Leutwyler [4]. For the p^6 Lagrangian more than a hundred counterterms were found [5]. In a more recent analysis [6] the number of independent counterterms was reduced to 90. Unfortunately, there is no simple prescription how to translate from one base into the other. The series of calculations on form factors taken up by us begun before this paper was published. In order to preserve the continuity with our previous papers on order p^6 chiral perturbation theory, we stick to the convention of [5]. There is no harm in keeping more than the minimal number of operators, as it is, for all practical purposes, impossible to determine by the necessary independent experiments such a large number of counterterms (including their finite parts), be it 143 or 90. Relations between individual physical processes are manifest in any case, as demonstrated in [7] and in the discussion below, where a relation of a part of the K_{l3} form factor to the pion electromagnetic form factor is established. The K_{l3} decay amplitude has been calculated some time ago to $\mathcal{O}(p^4)$ [8] and more recently to $\mathcal{O}(p^4, (m_d - m_u)p^2, e^2p^2)$ [9]. To $\mathcal{O}(p^6)$, the contribution of the double chiral logs has been calculated in [10].

We present here the results of a full p^6 and two-loop analysis of the semi-leptonic K_{l3} form factors, relying heavily on recent progress in the calculation of massive two-loop integrals [11,12]. Complementary calculations to this order involve the vector two-point function [13], $\pi\pi$ scattering [14] and the K_{l4} form factors [15].

As the relevant part of the p^6 Lagrangian contains so many unknown parameters, one may question the useful-

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ness of such calculations. Here are a few arguments in favor.

(1) In a given class of experiments, such as the electromagnetic and weak form factors of the light mesons, only a limited number of renormalization constants enter and relations between amplitudes can be tested [7].
(2) The unknown constants enter only polynomially, and precision experiments could separate the unambiguous predictions.
(3) Knowledge of the exact low-energy functional form of an amplitude may be important for the experimental extraction of low-energy parameters such as charge radii.
(4) The results may be used in model calculations which predict the polynomial terms. These calculations can then be compared with experiment.
(5) The question of convergence of the chiral perturbation theory may be addressed.

2 The matrix element

K mesons can decay into a pion and a lepton pair via the following channels:

$$K^+(p_1) \to \pi^0(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu), \qquad (1)$$
$$K^0(p_1) \to \pi^-(p_2)\ell^+(p_\ell)\nu_\ell(p_\nu), \qquad (2)$$

and their charge conjugate modes. The symbol ℓ stands for e or μ. We work in the isospin symmetry limit ($m_u = m_d$), where all the hadronic K_{l3} decay matrix elements are equal. We therefore restrict the discussion in the following to K^0_{l3} decay.

In the standard model only the vector current $V_\mu = \bar{u}\gamma_\mu s$ contributes to K_{l3} decay, and the hadronic matrix element has therefore the general form

$$\begin{aligned}&\langle \pi^-(p_2) \mid \bar{u}\gamma_\mu s \mid K^0(p_1)\rangle \\ &= (p_1+p_2)_\mu f_+(q^2) + (p_1-p_2)_\mu f_-(q^2),\end{aligned} \qquad (3)$$

where $q = p_1 - p_2$. The q^2 dependence of the form factors is usually approximated by

$$f_\pm(q^2) = f_\pm(0)\cdot\left(1 + \lambda_\pm \frac{q^2}{m_\pi^2}\right). \qquad (4)$$

The experimental method for the determination of $\lambda_\pm$ consists in comparing the measured q^2 distribution with a simulation using a constant form factor ($\lambda_\pm = 0$). This approximation could possibly be too crude for future accurate data.

The slope λ_+ has been remeasured recently in the CPLEAR experiment [2] with the result

$$\lambda_+ = 0.0245 \pm 0.0012_{\text{stat}} \pm 0.0022_{\text{syst}}. \qquad (5)$$

This value differs by almost two standard deviations from the previous world average [17] of

$$\lambda_+ = 0.0300 \pm 0.0026 \qquad (6)$$

(which is based on old data of the seventies) and from the prediction of order p^4 chiral perturbation theory [8,1].

The slope λ_- can only be measured in $K_{\mu 3}$ decay, and its status is even more controversial. It is common to consider also the so-called scalar form factor (because it specifies the S-wave projection of the crossed channel matrix element)

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_-(q^2). \qquad (7)$$

3 The Lagrangian of chiral perturbation theory

In the usual formulation of chiral perturbation theory the pseudo-scalar fields are collected in a unitary 3×3 matrix

$$U(x) = \exp\left[\mathrm{i}\frac{\Phi(x)}{F}\right], \qquad (8)$$

where F absorbs the dimensional dependence of the fields, and, in the chiral limit, is equal to the pion decay constant, $F = 92.4$ MeV. The 3×3 matrix Φ is given by

$$\Phi = \lambda_a\phi_a = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}, \qquad (9)$$

where λ_a are the Gell-Mann matrices.

An explicit breaking of chiral symmetry is introduced via the mass matrix

$$\chi = \begin{pmatrix} m_\pi^2 & 0 & 0 \\ 0 & m_\pi^2 & 0 \\ 0 & 0 & 2m_K^2 - m_\pi^2 \end{pmatrix}, \qquad (10)$$

where m_π and m_K are the unrenormalized masses of the π and K mesons. The mass of the η meson is given to this order by the Gell-Mann–Okubo relation

$$m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2. \qquad (11)$$

The mass term is related to the quark masses by $\chi = \text{const}\cdot\text{diag}(m_u, m_d, m_s)$ with $m_u = m_d$. To calculate form factors, we have to include the interaction with external boson fields. This is done by introducing gauge fields l_μ, r_μ:

$$l_\mu = \sum_{a=1}^{8} T_a l_\mu^a \quad \text{and} \quad r_\mu = \sum_{a=1}^{8} T_a r_\mu^a \qquad (12)$$

($T_a = \lambda_a/2$) with their field tensors

$$L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - \mathrm{i}[l_\mu, l_\nu], \qquad (13)$$
$$R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - \mathrm{i}[r_\mu, r_\nu], \qquad (14)$$

and replacing the usual derivative by a covariant one,

$$\partial_\mu U \to D_\mu U = \partial_\mu U + \mathrm{i} U l_\mu - \mathrm{i} r_\mu U. \quad (15)$$

In this way we have extended the global chiral $SU(3) \times SU(3)$ to a local symmetry. In case of the weak interaction, the external boson is the W and is given by

$$l_\mu = -\frac{e}{\sqrt{2}\sin\theta_{\mathrm{W}}}(W_\mu^+ T + \mathrm{h.c.}), \quad (16)$$

$$r_\mu = 0, \quad (17)$$

with

$$T = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (18)$$

With the building blocks D_μ, U, $U^\dagger$, $L_{\mu\nu}$, $R_{\mu\nu}$ and χ we can construct the Lagrangian of chiral perturbation theory

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots, \quad (19)$$

where $\mathcal{L}^{(2n)}$ denotes the most general expression with $2n$ powers of mass or covariant derivatives that is consistent with the symmetries of QCD. For the two lowest orders the result is [3] [4]

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\mathrm{Tr}(D_\mu U D^\mu U^\dagger) + \frac{F^2}{4}\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger), \quad (20)$$

$$\begin{aligned}\mathcal{L}^{(4)} &= L_1\{\mathrm{Tr}(D_\mu U D^\mu U^\dagger)\}^2 \\ &+ L_2 \mathrm{Tr}(D_\mu U D_\nu U^\dagger)\mathrm{Tr}(D^\mu U D^\nu U^\dagger) \\ &+ L_3 \mathrm{Tr}(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger) \\ &+ L_4 \mathrm{Tr}(D_\mu U D^\mu U^\dagger)\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger) \\ &+ L_5 \mathrm{Tr}(D_\mu U D^\mu U^\dagger(\chi U^\dagger + U\chi^\dagger)) \\ &+ L_6\{\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)\}^2 + L_7\{\mathrm{Tr}(\chi^\dagger U - U^\dagger\chi)\}^2 \\ &+ L_8 \mathrm{Tr}(\chi U^\dagger \chi U^\dagger + U\chi^\dagger U\chi^\dagger) \\ &- \mathrm{i}L_9 \mathrm{Tr}(L_{\mu\nu} D^\mu U D^\nu U^\dagger + R_{\mu\nu} D^\mu U^\dagger D^\nu U) \\ &+ L_{10}\mathrm{Tr}(L_{\mu\nu} U R^{\mu\nu} U^\dagger). \end{aligned} \quad (21)$$

The so-called low-energy constants $L_1, \ldots, L_{10}$ are unrenormalized coupling constants which must be determined by comparison with experiment. The $\mathcal{O}(p^6)$ Lagrangian was determined in [5]. Out of the 143 terms, we reproduce here only those which are relevant to semileptonic K decays:

$$\begin{aligned} F^2\mathcal{L}^{(6)}_{\mathrm{rel}} &= \beta_8 \mathrm{Tr}([D_\mu D_\nu U]_{\mathrm{m}}\{[D^\mu U]_{\mathrm{m}}, [D^\nu \chi]_{\mathrm{p}}\}) \quad (22) \\ &+ \beta_{14}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}(\{[D^\mu\chi]_{\mathrm{p}}, [\chi]_{\mathrm{m}}\} \\ &\quad -\{[D^\mu\chi]_{\mathrm{m}}, [\chi]_{\mathrm{p}}\})) \\ &+ \beta_{15}\{\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D^\mu\chi]_{\mathrm{p}})\mathrm{Tr}([\chi]_{\mathrm{m}}) \\ &\quad -\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[\chi]_{\mathrm{p}})\mathrm{Tr}([D^\mu\chi]_{\mathrm{m}})\} \\ &+ \beta_{16}\{\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D^\mu\chi]_{\mathrm{m}})\mathrm{Tr}([\chi]_{\mathrm{p}}) \\ &\quad -\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[\chi]_{\mathrm{m}})\mathrm{Tr}([D^\mu\chi]_{\mathrm{p}})\} \\ &+ \beta_{17}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D^\mu U]_{\mathrm{m}}[\chi]_{\mathrm{p}}[\chi]_{\mathrm{p}}) \\ &+ \beta_{18}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D^\mu U]_{\mathrm{m}}[\chi]_{\mathrm{p}})\mathrm{Tr}([\chi]_{\mathrm{p}}) \\ &+ \beta_{19}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D^\mu U]_{\mathrm{m}})\mathrm{Tr}([\chi]_{\mathrm{p}}[\chi]_{\mathrm{p}}) \\ &+ \beta_{20}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[\chi]_{\mathrm{p}})\mathrm{Tr}([D^\mu U]_{\mathrm{m}}[\chi]_{\mathrm{p}}) \\ &+ \beta_{21}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D^\mu U]_{\mathrm{m}})\mathrm{Tr}([\chi]_{\mathrm{p}})\mathrm{Tr}([\chi]_{\mathrm{p}}) \\ &+ \mathrm{i}\beta_{22}\mathrm{Tr}([D_\mu D^\rho U]_{\mathrm{m}}[[D_\rho U]_{\mathrm{m}}, [D_\nu G^{\mu\nu}]_{\mathrm{p}}]) \\ &+ \mathrm{i}\beta_{23}\mathrm{Tr}([D_\mu D^\rho U]_{\mathrm{m}}[[D_\nu U]_{\mathrm{m}}, [D_\rho G^{\mu\nu}]_{\mathrm{p}}]) \\ &+ \mathrm{i}\beta_{24}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D_\nu U]_{\mathrm{m}}\{[\chi]_{\mathrm{p}}, [G^{\mu\nu}]_{\mathrm{p}}\}) \\ &+ \mathrm{i}\beta_{25}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[\chi]_{\mathrm{p}}[D_\nu U]_{\mathrm{m}}[G^{\mu\nu}]_{\mathrm{p}}) \\ &+ \mathrm{i}\beta_{26}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}[D_\nu U]_{\mathrm{m}}[G^{\mu\nu}]_{\mathrm{p}})\mathrm{Tr}([\chi]_{\mathrm{p}}) \\ &+ \mathrm{i}\beta_{27}\mathrm{Tr}([D_\mu U]_{\mathrm{m}}([[D_\nu\chi]_{\mathrm{m}}, [G^{\mu\nu}]_{\mathrm{p}}] \\ &\quad -[[\chi]_{\mathrm{m}}, [D_\nu G^{\mu\nu}]_{\mathrm{p}}])), \end{aligned}$$

where we used the notation of [5]

$$G_{\mu\nu} = R_{\mu\nu} U + U L_{\mu\nu}, \quad (23)$$

$$[A]_{\mathrm{m}} = \frac{1}{2}(AU^\dagger - UA^\dagger), \quad (24)$$

$$[A]_{\mathrm{p}} = \frac{1}{2}(AU^\dagger + UA^\dagger), \quad (25)$$

with a slight change of notation for the couplings

$$\beta_i \doteq F^2 B_i \quad \text{(Fearing–Scherer)} \quad (26)$$

so as to use dimensionless quantities.

The Feynman rules can be derived by expanding $U = \exp(\mathrm{i}\Phi/F)$ everywhere in $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)}$ and identifying the relevant vertex monomials in $\mathcal{L}$. Before discussing the Feynman rules in detail in the next section, we would like to make a remark on the definition of the form factors. The currents entering in (3) are defined on the quark level. The connection to the effective theory is established by identifying these currents with the Noether currents of the chiral symmetry,

$$\bar{u}\gamma_\mu s = V_{\mu,4} - \mathrm{i}V_{\mu,5}, \quad (27)$$

where $V_a^\mu = l_a^\mu + r_a^\mu$, $a = 1, \ldots, 8$, denotes the vector current in the effective theory.

It is obviously necessary to distinguish also graphically the vertices from $\mathcal{L}^{(2)}$, $\mathcal{L}^{(4)}$ and $\mathcal{L}^{(6)}$. We use the conventions: a filled circle ● stands for a vertex from $\mathcal{L}^{(2)}$, a filled square ■ for a vertex from $\mathcal{L}^{(4)}$ and an open square □ for a vertex from $\mathcal{L}^{(6)}$.

3.1 Pure meson vertices

If loop corrections to form factor diagrams are considered various vertices enter which involve only mesons. In the form factor calculation to $\mathcal{O}(p^6)$ five vertices from $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)}$ will contribute.

From $\mathcal{L}^{(2)}$ we have to consider vertices with 4 and 6 meson fields:

$$48F^2\mathcal{L}^{(2)}_{\mathrm{four}} = 2\mathrm{Tr}([\phi, \partial_\mu\phi]\phi\partial^\mu\phi) + \mathrm{Tr}(\chi\phi^4) \quad (28)$$

and

$$1440F^4\mathcal{L}^{(2)}_{\mathrm{six}} = 2\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi\phi^4) - 8\mathrm{Tr}(\partial_\mu\phi\phi\partial^\mu\phi\phi^3)$$

$$+ 6\mathrm{Tr}(\partial_\mu\phi\phi^2\partial^\mu\phi\phi^2) - \mathrm{Tr}(\chi\phi^6). \tag{29}$$

Similarly we have the two- and four-meson vertex from $\mathcal{L}^{(4)}$:

$$\begin{aligned}F^2\mathcal{L}^{(4)}_{\mathrm{two}} &= 2L_4\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi)\mathrm{Tr}(\chi) \\ &+ 2L_5\mathrm{Tr}(\chi\partial_\mu\phi\partial^\mu\phi) - 4L_6\mathrm{Tr}(\chi\phi^2)\mathrm{Tr}(\chi) \\ &- 4L_7\{\mathrm{Tr}(\chi\phi)\}^2 - 2L_8\{\mathrm{Tr}(\chi^2\phi^2) + \mathrm{Tr}(\chi\phi\chi\phi)\},\end{aligned} \tag{30}$$

$$\begin{aligned}6F^4\mathcal{L}^{(4)}_{\mathrm{four}} &= 6L_1\{\mathrm{Tr}(\partial_\nu\phi\partial^\nu\phi)\}^2 \\ &+ 6L_2\mathrm{Tr}(\partial_\mu\phi\partial_\nu\phi)\mathrm{Tr}(\partial^\mu\phi\partial^\nu\phi) \\ &+ 6L_3\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi\partial_\nu\phi\partial^\nu\phi) \\ &- 2L_4\{\mathrm{Tr}([\phi,\partial_\nu\phi]\phi\partial^\nu\phi)\mathrm{Tr}(\chi) \\ &+ 3\mathrm{Tr}(\partial_\nu\phi\partial^\nu\phi)\mathrm{Tr}(\chi\phi^2)\} \\ &- L_5\{2\mathrm{Tr}(\chi\phi^2\partial_\nu\phi\partial^\nu\phi) + 3\mathrm{Tr}(\chi\phi\partial_\nu\phi\partial^\nu\phi\phi) \\ &- \mathrm{Tr}(\chi\phi\partial_\nu\phi\phi\partial^\nu\phi) + \mathrm{Tr}(\chi\partial_\nu\phi\phi^2\partial^\nu\phi) \\ &- \mathrm{Tr}(\chi\partial_\nu\phi\phi\partial^\nu\phi\phi) + 2\mathrm{Tr}(\chi\partial_\nu\phi\partial^\nu\phi\phi^2)\} \\ &+ 2L_6\{\mathrm{Tr}(\chi\phi^4)\mathrm{Tr}(\chi) + 3[\mathrm{Tr}(\chi\phi^2)]^2\} \\ &+ 8L_7\mathrm{Tr}(\chi\phi^3)\mathrm{Tr}(\chi\phi) \\ &+ L_8\{\mathrm{Tr}(\chi^2\phi^4) + 2\mathrm{Tr}(\chi\phi\chi\phi^3) + 3\mathrm{Tr}(\chi\phi^2\chi\phi^2) \\ &+ 2\mathrm{Tr}(\chi\phi^3\chi\phi)\}.\end{aligned} \tag{31}$$

The parameters $L_1, \ldots, L_{10}$, the so-called low-energy constants of $\mathcal{L}^{(4)}$, are not fixed by the symmetries but must be determined by comparing perturbative results with experimental data or with models. The low-energy constants also serve to renormalize the loop diagrams. Therefore they contain divergent pieces which, in dimensional regularization, manifest themselves in $1/\varepsilon$-poles ($D = 4 - 2\varepsilon$) and which were calculated in [4].

Finally, the two-meson vertex from $\mathcal{L}^{(6)}$ enters:

$$\begin{aligned}F^4\mathcal{L}^{(6)}_{\mathrm{two}} &= -\beta_{17}\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi\chi\chi) \\ &- \beta_{18}\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi\chi)\mathrm{Tr}(\chi) \\ &- \beta_{19}\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi)\mathrm{Tr}(\chi\chi) \\ &- \beta_{20}\mathrm{Tr}(\partial_\mu\phi\chi)\mathrm{Tr}(\partial^\mu\phi\chi) \\ &- \beta_{21}\mathrm{Tr}(\partial_\mu\phi\partial^\mu\phi)\mathrm{Tr}(\chi)\mathrm{Tr}(\chi).\end{aligned} \tag{32}$$

3.2 W boson–meson vertices

Every diagram contributing to K_{l3} decay contains one vertex where the external W boson couples to the mesons. The Feynman rules of the corresponding vertices result from the terms in $\mathcal{L}$ that are linear in the gauge fields. Thus, the left-handed and right-handed mesonic currents that couple to the external pseudo-scalar mesons are given by

$$J^L_{\mu,a} = \left.\frac{\delta\mathcal{L}}{\delta l^{\mu,a}}\right|_{r_\mu=l_\mu=0}, \quad J^R_{\mu,a} = \left.\frac{\delta\mathcal{L}}{\delta r^{\mu,a}}\right|_{r_\mu=l_\mu=0}. \tag{33}$$

The result for the currents from $\mathcal{L}^{(2)}$ reads

$$\begin{aligned}J^L_{\mu,a}[\mathcal{L}^{(2)}] &= \frac{\mathrm{i}}{4}\mathrm{Tr}(T_a[\partial_\mu\phi,\phi]) \\ &- \frac{\mathrm{i}}{48F^2}\mathrm{Tr}T_a([\partial_\mu\phi,\phi^3] - 3\phi[\partial_\mu\phi,\phi]\phi) \\ &+ \frac{\mathrm{i}}{1440F^4}\mathrm{Tr}T_a([\partial_\mu\phi,\phi^5] - 5\phi[\partial_\mu\phi,\phi^3]\phi \\ &\quad +10\phi^2[\partial_\mu\phi,\phi]\phi^2),\end{aligned} \tag{34}$$

where we have only written the terms contributing to K_{l3} decay at $\mathcal{O}(p^6)$. We represent a W vertex from $\mathcal{L}^{(2)}$ by a filled circle ●.

The result for the relevant terms of the current from $\mathcal{L}^{(4)}$ reads

$$\begin{aligned}F^2J^L_{\mu,a}[\mathcal{L}^{(4)}]_{\mathrm{two}} &= 2\mathrm{i}L_4\mathrm{Tr}(T_a[\partial_\mu\phi,\phi])\mathrm{Tr}(\chi) \\ &+ \mathrm{i}L_5\mathrm{Tr}T_a(\chi\partial_\mu\phi\phi + \partial_\mu\phi\chi\phi - \phi\chi\partial_\mu\phi - \phi\partial_\mu\phi\chi) \\ &- \mathrm{i}L_9\partial^\nu\mathrm{Tr}(T_a[\partial_\mu\phi,\partial_\nu\phi])\end{aligned} \tag{35}$$

for the two-meson–W vertex and

$$\begin{aligned}&12F^4J^L_{\mu,a}[\mathcal{L}^{(4)}]_{\mathrm{four}} \\ &= 24\mathrm{i}L_1\mathrm{Tr}(T_a[\partial_\mu\phi,\phi])\mathrm{Tr}(\partial_\nu\phi\partial^\nu\phi) \\ &+ 24\mathrm{i}L_2\mathrm{Tr}(T_a[\partial^\nu\phi,\phi])\mathrm{Tr}(\partial_\mu\phi\partial_\nu\phi) \\ &+ 12\mathrm{i}L_3\mathrm{Tr}T_a(\{\partial_\mu\phi,\partial_\nu\phi\partial^\nu\phi\}\phi - \phi\{\partial_\mu\phi,\partial_\nu\phi\partial^\nu\phi\}) \\ &- 2\mathrm{i}L_4\{6\mathrm{Tr}(T_a[\partial_\mu\phi,\phi])\mathrm{Tr}(\chi\phi^2) + \mathrm{Tr}T_a([\partial_\mu\phi,\phi^3] \\ &- 3\phi[\partial_\mu\phi,\phi]\phi)\mathrm{Tr}(\chi)\} \\ &- \mathrm{i}L_5\{\mathrm{Tr}T_a(\chi[\partial_\mu\phi,\phi]\phi^2 + 2\chi\phi^2\partial_\mu\phi\phi - 2\phi\chi\{\partial_\mu\phi,\phi^2\}) \\ &+ \mathrm{Tr}T_a(2\partial_\mu\phi\chi\phi^3 + 4\phi\chi\phi\partial_\mu\phi\phi + 3[\partial_\mu\phi,\phi]\chi\phi^2) \\ &+ \mathrm{Tr}T_a(3\phi^2\chi[\partial_\mu\phi,\phi] + 2\{\partial_\mu\phi,\phi^2\}\chi\phi - 4\phi\partial_\mu\phi\phi\chi\phi) \\ &+ \mathrm{Tr}T_a(-2\phi^3\chi\partial_\mu\phi - 2\phi\partial_\mu\phi\phi^2\chi + \phi^2[\partial_\mu\phi,\phi]\chi)\} \\ &+ \mathrm{i}L_9\partial^\nu\{\mathrm{Tr}T_a(-3\phi[\partial_\mu\phi,\partial_\nu\phi]\phi + \partial_\mu\phi\phi^2\partial_\nu\phi - \partial_\nu\phi\phi^2\partial_\mu\phi) \\ &+ \mathrm{Tr}T_a(2\phi^2[\partial_\mu\phi,\partial_\nu\phi] + 2[\partial_\mu\phi,\partial_\nu\phi]\phi^2 \\ &- [\partial_\mu\phi\phi,\partial_\nu\phi\phi] - \mathrm{Tr}T_a([\phi\partial_\mu\phi,\phi\partial_\nu\phi])\}\end{aligned} \tag{36}$$

for the four-meson vertex with one W boson. In the diagrams we represent a W vertex from $\mathcal{L}^{(4)}$ by a filled square ■.

Finally a two-meson-W vertex from $\mathcal{L}^{(6)}$ contributes

$$\begin{aligned}4F^4J^L_{\mu,a}[\mathcal{L}^{(6)}] &= 2\mathrm{i}\beta_8\mathrm{Tr}T_a[\chi,\{\partial_\mu\partial_\nu\phi,\partial^\nu\phi\}] \\ &+ \mathrm{i}\beta_{14}\mathrm{Tr}T_a(2\chi\phi\partial_\mu\phi\chi - 2\chi\partial_\mu\phi\phi\chi + \chi\chi\partial_\mu\phi\phi \\ &- \chi\chi\phi\partial_\mu\phi + \partial_\mu\phi\phi\chi\chi - \phi\partial_\mu\phi\chi\chi) \\ &+ 2\mathrm{i}\beta_{15}\{\mathrm{Tr}(\chi\phi)\mathrm{Tr}T_a[\partial_\mu\phi,\chi] - \mathrm{Tr}(\chi\partial_\mu\phi)\mathrm{Tr}T_a[\phi,\chi]\} \\ &+ \mathrm{i}\beta_{16}\mathrm{Tr}(\chi)\mathrm{Tr}T_a[\{\partial_\mu\phi,\phi\},\chi] \\ &+ 2\mathrm{i}\beta_{17}\mathrm{Tr}T_a(\phi\partial_\mu\phi\chi\chi - \partial_\mu\phi\chi\chi\phi + \phi\chi\chi\partial_\mu\phi - \chi\chi\partial_\mu\phi\phi) \\ &+ 2\mathrm{i}\beta_{18}\mathrm{Tr}(\chi)\mathrm{Tr}T_a(\phi\partial_\mu\phi\chi - \partial_\mu\phi\chi\phi + \phi\chi\partial_\mu\phi - \chi\partial_\mu\phi\phi) \\ &+ 4\mathrm{i}\beta_{19}\{\mathrm{Tr}(\chi\chi)\mathrm{Tr}(T_a[\phi,\partial_\mu\phi]) \\ &+ 2\mathrm{Tr}(\chi[\chi,\phi])\mathrm{Tr}(T_a\partial_\mu\phi)\} \\ &+ 4\mathrm{i}\beta_{20}\mathrm{Tr}(\chi\partial_\mu\phi)\mathrm{Tr}(T_a[\phi,\chi]) \\ &+ 4\mathrm{i}\beta_{21}\mathrm{Tr}(\chi)\mathrm{Tr}(\chi)\mathrm{Tr}(T_a[\phi,\partial_\mu\phi]) \\ &+ 4\mathrm{i}\beta_{22}\mathrm{Tr}T_a(\partial_\nu\partial^\nu[\partial_\mu\partial_\rho\phi,\partial^\rho\phi] - \partial_\mu\partial^\nu[\partial_\nu\partial_\rho\phi,\partial^\rho\phi]) \\ &+ 4\mathrm{i}\beta_{23}\partial^\nu\partial^\rho\mathrm{Tr}T_a([\partial_\mu\partial_\rho\phi,\partial_\nu\phi] - [\partial_\nu\partial_\rho\phi,\partial_\mu\phi]) \\ &- 4\mathrm{i}\beta_{24}\partial^\nu\mathrm{Tr}([\partial_\mu\phi,\partial_\nu\phi]\{\chi,T_a\}) \\ &- 4\mathrm{i}\beta_{25}\partial^\nu\mathrm{Tr}T_a(\partial_\mu\phi\chi\partial_\nu\phi - \partial_\nu\phi\chi\partial_\mu\phi)\end{aligned}$$

$$
\begin{aligned}
&- 4\mathrm{i}\beta_{26}\partial^\nu \mathrm{Tr}(T_a[\partial_\mu\phi, \partial_\nu\phi])\mathrm{Tr}(\chi) \\
&+ 2\mathrm{i}\beta_{27}\partial^\nu \mathrm{Tr}T_a([\partial_\mu\phi, \{\partial_\nu\phi, \chi\}] - [\partial_\nu\phi, \{\partial_\mu\phi, \chi\}]).
\end{aligned}
$$

A W vertex from $\mathcal{L}^{(6)}$ is represented by an open square $\square$. From the interactions given above, the Feynman rules can be extracted by transforming into momentum space and symmetrizing over the meson fields.

3.3 Renormalization scheme

Before evaluating the loop diagrams, we must specify the regularization and renormalization scheme. In our calculation we are using dimensional regularization and the so-called *GL* scheme which is defined in the following way: each diagram of order $\mathcal{O}(p^{2n})$ is multiplied with a factor $\mathrm{e}^{(1-n)\alpha(\varepsilon)}$ where $D = 4-2\varepsilon$ is the dimension of space-time and $\alpha(\varepsilon)$ is given by

$$
(4\pi)^\varepsilon \Gamma(-1+\varepsilon) = -\frac{\mathrm{e}^{\alpha(\varepsilon)}}{\varepsilon}, \tag{38}
$$

that is

$$
\alpha(\varepsilon) = \varepsilon(1-\gamma+\log 4\pi) + \varepsilon^2\left(\frac{\pi^2}{12}+\frac{1}{2}\right) + \mathcal{O}(\varepsilon^3). \tag{39}
$$

Because of $\alpha(0) = 0$ the total $\mathcal{O}(p^6)$ result is unchanged in $D = 4$ dimensions. The reason for this modification of each diagram is to eliminate the geometric factor $(4\pi)^\varepsilon\Gamma(-1+\varepsilon)$ appearing in the one-loop integrals. This renormalization scheme is very similar to the well-known $\overline{\mathrm{MS}}$ scheme, where each diagram is multiplied by a factor $(4\pi)^{-\varepsilon}\mathrm{e}^{\gamma\varepsilon}$ per loop (instead of $\mathrm{e}^{-\alpha(\varepsilon)}$).

The GL scheme extends the usual one-loop scheme introduced by Gasser and Leutwyler [4] in a natural way. This can be understood by considering the renormalization constants L_i of $\mathcal{L}^{(4)}$: In D-dimensional space-time they have dimension $D-4$ and their dimension is made manifest by the mass scale μ of dimensional regularization:

$$
L_i = \mu^{D-4}L_i(\mu, D). \tag{40}
$$

$L_i(\mu, D)$ has the same μ dependence as a one-loop integral, because L_i itself is independent of μ. It can be expanded in a Laurent series around $\varepsilon = 0$ in the same way as a one-loop integral:

$$
L_i(\mu, D) = \frac{L_i^{(-1)}}{\varepsilon} + L_i^{(0)}(\mu) + \varepsilon L_i^{(1)}(\mu) + \mathcal{O}(\varepsilon^2). \tag{41}
$$

In the usual one-loop scheme one chooses

$$
L_i^{(-1)} = -\frac{\Gamma_i}{32\pi^2}, \tag{42}
$$

$$
L_i^{(0)}(\mu) = L_i^{\mathrm{ren}}(\mu) - \frac{\Gamma_i}{32\pi^2}[1-\gamma+\log 4\pi], \tag{43}
$$

where Γ_i are numbers which can be found in [4]. The second term in $L_i^{(0)}$ is constructed so that it cancels in the ε^0 coefficient after multiplication with the GL factor $\mathrm{e}^{-\alpha(\varepsilon)}$:

$$
\begin{aligned}
L_i^{\mathrm{GL}}(\mu, D) &= \mathrm{e}^{-\alpha(\varepsilon)}L_i(\mu, D) \qquad (44)\\
&= \frac{L_i^{(-1)}}{\varepsilon} + L_i^{(0),\mathrm{GL}}(\mu) + \varepsilon L_i^{(1),\mathrm{GL}}(\mu) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

with $L_i^{(0),\mathrm{GL}}(\mu) = L_i^{\mathrm{ren}}(\mu)$.

The dimension of the $\mathcal{L}^{(6)}$ parameters β_i appearing in (32) can be treated in the same way:

$$
\beta_i = \mu^{2D-8}\beta_i(\mu, D), \tag{45}
$$

where $\beta_i(\mu, D)$ behaves like a two-loop integral. Its Laurent series in the above GL scheme is given by

$$
\begin{aligned}
\beta_i^{\mathrm{GL}}(\mu, D) &= \mathrm{e}^{-2\alpha(\varepsilon)}\beta_i(\mu, D) \qquad (46)\\
&= \frac{\beta_i^{(-2)}}{\varepsilon^2} + \frac{\beta_i^{(-1),\mathrm{GL}}(\mu)}{\varepsilon} + \beta_i^{(0),\mathrm{GL}}(\mu) + \mathcal{O}(\varepsilon).
\end{aligned}
$$

3.4 Mass and wavefunction renormalization

To order p^6, finite S-matrix elements in chiral perturbation theory are obtained by multiplying the unrenormalized one-particle irreducible (1PI) Feynman diagrams obtained from $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)}$ with a factor $Z^{1/2}$ per external meson, where Z is the wavefunction renormalization constant for this meson. To be on familiar ground, we start by calculating the mass and wavefunction renormalization from the renormalized propagator

$$
\frac{\mathrm{i}}{p^2 - m^2 - \Sigma(p^2)}, \tag{47}
$$

where m denotes the bare meson mass and $\Sigma(p^2)$ the 1PI unrenormalized self-energy which is given perturbatively by

$$
\Sigma(p^2) = \Sigma_1(p^2) + \Sigma_2(p^2) + \ldots \tag{48}
$$

The leading $\mathcal{O}(p^2)$ contribution vanishes, so that Σ_1 , Σ_2 represent the contributions of $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$.

From the condition that the renormalized propagator develops a pole with residue 1 at $p^2 = m_{\mathrm{ph}}^2$, where m_{ph} is the physical or pole mass, one derives the conditions

$$
\delta m^2 = m_{\mathrm{ph}}^2 - m^2 = \Sigma(p^2 = m_{\mathrm{ph}}^2, m^2, F), \tag{49}
$$

$$
Z^{-1} = 1 - \Sigma'(p^2 = m_{\mathrm{ph}}^2, m^2, F), \tag{50}
$$

where m stands symbolically for the set of unrenormalized masses and F is the unrenormalized pion decay constant (all assumed to be given as functions of the renormalized or physical parameters). Perturbatively, Z is therefore given by

$$
Z = 1 + \delta Z_1 + \delta Z_2 + \ldots \tag{51}
$$

with the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ corrections

$$
\delta Z_1 = \Sigma_1'(p^2 = m^2, m^2, F), \tag{52}
$$

$$
\delta Z_2 = \Sigma_2'(m_{\mathrm{ph}}^2) + \Sigma_1'(m_{\mathrm{ph}}^2)^2, \tag{53}
$$

where we have expanded Σ_1' around $p^2 = m^2$ and used the fact that the term involving the second derivative of $\Sigma_1(p^2)$ vanishes, i.e. Σ_1' is independent of p^2. In (52) the unrenormalized quantities m^2, F must be expressed by their physical values according to (49) and (60) below.

Each external meson propagator must be multiplied by a factor

$$\sqrt{Z} = \sqrt{1+\delta Z} = 1 + \frac{\delta Z_1}{2} + \left[\frac{\delta Z_2}{2} - \frac{(\delta Z_1)^2}{8}\right] + \mathcal{O}(p^6). \tag{54}$$

We have to calculate $\Sigma(p^2)$ for π, K mesons in order to determine Z_π, Z_K and δm_π^2, δm_K^2. The $\mathcal{O}(p^4)$ results are well known [4]:

$$\delta Z_1^\pi = -\frac{1}{3F^2}\{A_0(m_K^2) + 2A_0(m_\pi^2) + 24L_4(2m_K^2 + m_\pi^2) + 24L_5 m_\pi^2\}, \tag{55}$$

$$\delta Z_1^K = -\frac{1}{4F^2}\{A_0(m_\eta^2) + 2A_0(m_K^2) + A_0(m_\pi^2) + 32L_4(2m_K^2 + m_\pi^2) + 32L_5 m_K^2\}, \tag{56}$$

$$\delta m_\pi^2 = \frac{1}{6F^2}\{m_\pi^2 A_0(m_\eta^2) - 3m_\pi^2 A_0(m_\pi^2) - 48m_\pi^2(2m_K^2 + m_\pi^2)L_4 - 48L_5 m_\pi^4 + 96L_6 m_\pi^2(2m_K^2 + m_\pi^2) + 96L_8 m_\pi^4\}, \tag{57}$$

$$\delta m_K^2 = \frac{1}{12F^2}\Big\{ -4m_K^2 A_0(m_\eta^2) - 96L_4 m_K^2(2m_K^2 + m_\pi^2) - 96L_5 m_K^4 + 192L_6 m_K^2(2m_K^2 + m_\pi^2) + 192L_8 m_K^4\Big\}. \tag{58}$$

The function $A_0(m^2)$ is the standard tadpole integral

$$A_0(m^2) = \mu^{4-D}\int \frac{\mathrm{d}^D k}{\mathrm{i}(2\pi)^D}\frac{1}{k^2 - m^2}, \tag{59}$$

where $D = 4 - 2\varepsilon$ is the dimension of space-time.

In addition, we need the renormalization of the pion decay constant

$$\delta F = F_\pi - F, \tag{60}$$

where F_π is the physical pion decay constant. We only quote the result [4]:

$$\delta F = \frac{1}{2F}\{A_0(m_K^2) + 2A_0(m_\pi^2) + 8L_4(2m_K^2 + m_\pi^2) + 8L_5 m_\pi^2\}. \tag{61}$$

It should be noted that the renormalization constants δm^2 and δF defined above are finite. The divergences and scale dependence of the loop integrals are canceled by similar factors in the counterterms L_i from $\mathcal{L}^{(4)}$.

The self-energy diagrams contributing up to two loops and to order p^6 are given in Fig. 1. External legs are fixed by the mesons considered, while one has to sum over all possible internal meson lines.

We call a two-loop diagram "reducible", if the two-loop integrations decouple, i.e. if they are given by a product of one-loop integrals. Otherwise they are called "irreducible". The $\mathcal{O}(p^6)$ correction of the wavefunction renormalization Z consists of three parts:

$$\delta Z_2 = \delta Z_2^{\mathrm{red}} + \delta Z_2^{\mathrm{irred}} + \delta Z_2[\mathcal{L}^{(6)}], \tag{62}$$

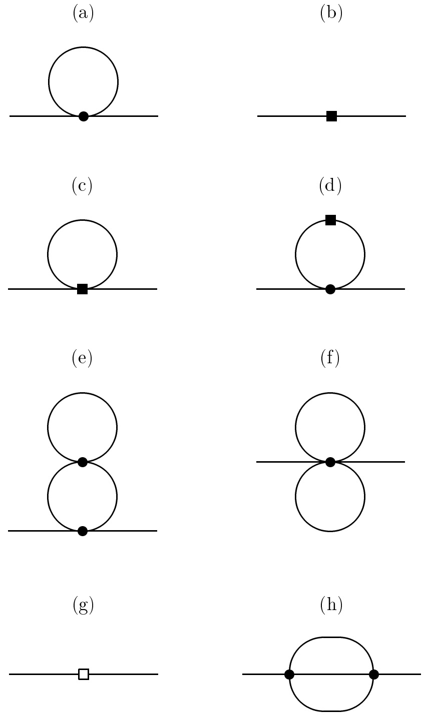

Fig. 1a–h. Self-energy diagrams up to order p^6. $\mathcal{L}^{(2)}$ vertices are denoted by filled circles (●), $\mathcal{L}^{(4)}$ vertices by filled squares (■), and an $\mathcal{L}^{(6)}$ vertex by an open square (□)

which are given below for π and K.

The one-loop and the reducible two-loop diagrams of Fig. 1 yield for the pion wavefunction renormalization Z^π an $\mathcal{O}(p^6)$ contribution

$$\begin{aligned}\delta Z_2^{\pi,\mathrm{red}} = \frac{1}{180F^4}\Big\{ & 2880\big[2L_1 + 4L_2 + L_3\big]A_2(m_\pi^2) \\ & + 2880\big[4L_2 + L_3\big]A_2(m_K^2) + 960\big[3L_2 + L_3\big]A_2(m_\eta^2) \\ & + 960m_\pi^2\big[(2m_K^2 + m_\pi^2)(L_4 - 2L_6) + m_\pi^2(L_5 - 2L_8)\big] \\ & \quad \times A_{0;2}(m_\pi^2) \\ & + 480m_K^2\big[(m_\pi^2 + 2m_K^2)(L_4 - 2L_6) + m_K^2(L_5 - 2L_8)\big] \\ & \quad \times A_{0;2}(m_K^2) \\ & + 720m_\pi^2\big[12L_1 + 4L_2 + 6L_3 - 6L_4 - 3L_5\big]A_0(m_\pi^2) \\ & + 480\big[24m_K^2 L_1 + 6m_K^2 L_3 - 12m_K^2 L_4 - (m_\pi^2 + 2m_K^2)L_5\big] \\ & \quad \times A_0(m_K^2) \\ & + 80\big[2(4m_K^2 - m_\pi^2)(6L_1 + L_3 - 3L_4) - 3m_\pi^2 L_5\big]A_0(m_\eta^2) \\ & + 20m_\pi^2\big[3A_0(m_\pi^2) - A_0(m_\eta^2)\big]A_{0;2}(m_\pi^2) \\ & + 20m_K^2 A_0(m_\eta^2)A_{0;2}(m_K^2) \\ & - 15A_0(m_K^2)A_0(m_\pi^2) + 6A_0(m_K^2)^2 \\ & + 9A_0(m_\eta^2)A_0(m_K^2)\Big\} + (\delta Z_1^\pi)^2, \end{aligned} \tag{63}$$

where δZ_1^π is the $\mathcal{O}(p^4)$ result from (55) and the additional functions $A_{0;2}$ and A_2 are related to the tadpole integral A_0 and the dimension D of space-time (cf. (108) and (104) in Appendix A):

$$A_{0;2}(m^2) = \frac{1}{m^2}\left(\frac{D}{2}-1\right)A_0(m^2) \text{ and} \qquad (64)$$
$$A_2(m^2) = \frac{m^2}{D}A_0(m^2).$$

For the irreducible two-loop contributions of Fig. 1, which involve higher transcendental functions, we only quote the exact result for the divergent part and a numerical result for the finite part. The latter involves an arbitrary scale μ which cancels in the final answer for the form factor. For the choice

$$\mu = m_\rho = 770\,\text{MeV}, \quad m_K = 495\,\text{MeV},$$
$$m_\pi = 0.28 m_K, \quad F_\pi = 92.4\,\text{MeV}, \qquad (65)$$

m_η^2 being an abbreviation for the Gell-Mann–Okubo term $(4/3)m_K^2 - (1/3)m_\pi^2$, and the definition

$$\text{Lg}(m^2) \doteq \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma$$

we obtain

$$\begin{aligned}\delta Z_2^{\pi,\text{irred}} &= 2.015180506 + \frac{78m_K^4 + 22m_K^2 m_\pi^2 + 89m_\pi^4}{72(4\pi)^4F^4\varepsilon^2}\\ &+ \frac{1}{288(4\pi)^4F^4\varepsilon}\Big\{1452m_K^4 - 36m_K^2m_\pi^2 + 1163m_\pi^4\\ &- 4(24m_K^4 + 14m_K^2m_\pi^2 - 5m_\pi^4)\text{Lg}(m_\eta^2)\\ &- 528m_K^4\text{Lg}(m_K^2)\\ &- 12m_\pi^2(10m_K^2 + 61m_\pi^2)\text{Lg}(m_\pi^2)\Big\}. \qquad (66)\end{aligned}$$

Similarly we obtain for the reducible part of the kaon wavefunction renormalization

$$\begin{aligned}\delta Z_2^{K,\text{red}} &= \frac{1}{360F^4}\Big\{4320\big[4L_2 + L_3\big]A_2(m_\pi^2) \qquad (67)\\ &+ 2880\big[4L_1 + 10L_2 + 3L_3\big]A_2(m_K^2)\\ &+ 480\big[12L_2 + L_3\big]A_2(m_\eta^2)\\ &+ 720m_\pi^2\big[(2m_K^2 + m_\pi^2)(L_4 - 2L_6) + m_\pi^2(L_5 - 2L_8)\big]\\ &\quad\times A_{0;2}(m_\pi^2)\\ &+ 1440m_K^2\big[(2m_K^2 + m_\pi^2)(L_4 - 2L_6) + m_K^2(L_5 - 2L_8)\big]\\ &\quad\times A_{0;2}(m_K^2)\\ &+ 80\big[(4m_K^2 - m_\pi^2)^2L_5 - 48(m_K^2 - m_\pi^2)^2L_7\\ &+ 3(4m_K^2 - m_\pi^2)(2m_K^2 + m_\pi^2)(L_4 - 2L_6)\\ &- 6(8m_K^4 - 8m_K^2m_\pi^2 + 3m_\pi^4)L_8\big]A_{0;2}(m_\eta^2)\\ &+ 720\big[6m_\pi^2(L_3 - 2L_4 + 4L_1) - (2m_\pi^2 + m_K^2)L_5\big]A_0(m_\pi^2)\\ &+ 1440m_K^2\big[16L_1 + 4L_2 + 6L_3 - 8L_4 - 3L_5\big]A_0(m_K^2)\\ &+ 80\big[2(4m_K^2 - m_\pi^2)(5L_3 - 6L_4 + 12L_1)\\ &+ 3(2m_\pi^2 - 7m_K^2)L_5\big]A_0(m_\eta^2)\\ &+ 45m_\pi^2A_0(m_\pi^2)A_{0;2}(m_\pi^2) - 45m_\pi^2A_0(m_\pi^2)A_{0;2}(m_\eta^2)\\ &+ 120m_K^2A_0(m_K^2)A_{0;2}(m_\eta^2) - 15m_\pi^2A_0(m_\eta^2)A_{0;2}(m_\pi^2)\\ &+ 60m_K^2A_0(m_\eta^2)A_{0;2}(m_K^2)\\ &+ 5(7m_\pi^2 - 16m_K^2)A_0(m_\eta^2)A_{0;2}(m_\eta^2)\\ &+ 45A_0(m_\pi^2)^2 - 18A_0(m_K^2)^2 - 27A_0(m_\eta^2)^2\\ &- 54A_0(m_\pi^2)A_0(m_\eta^2) - 27A_0(m_\pi^2)A_0(m_K^2)\\ &+ 81A_0(m_K^2)A_0(m_\eta^2)\Big\} + (\delta Z_1^K)^2.\end{aligned}$$

The irreducible diagram yields

$$\begin{aligned}\delta Z_2^{K,\text{irred}} &= 2.8602989531 + \frac{142m_K^4 + 5m_K^2m_\pi^2 + 42m_\pi^4}{72(4\pi)^4F^4\varepsilon^2}\\ &+ \frac{1}{288(4\pi)^4F^4\varepsilon}\Big\{1769m_K^4 + 186m_K^2m_\pi^2\\ &+ 624m_\pi^4 - 4(56m_K^4 - 26m_K^2m_\pi^2 + 3m_\pi^4)\text{Lg}(m_\eta^2)\\ &- 24m_K^2(38m_K^2 + 3m_\pi^2)\text{Lg}(m_K^2)\\ &- 36m_\pi^2(2m_K^2 + 9m_\pi^2)\text{Lg}(m_\pi^2)\Big\}. \qquad (68)\end{aligned}$$

Finally, the contribution arising from the $\mathcal{L}^{(6)}$ constants of (22) are

$$\begin{aligned}\delta Z_2^\pi[\mathcal{L}^{(6)}] &= \frac{1}{F^4}\Big\{4m_\pi^4\beta_{17} + 4m_\pi^2(2m_K^2 + m_\pi^2)\beta_{18} \qquad (69)\\ &+ 4(4m_K^4 - 4m_K^2m_\pi^2 + 3m_\pi^4)\beta_{19}\\ &+ 4(4m_K^4 + 4m_K^2m_\pi^2 + m_\pi^4)\beta_{21}\Big\},\\ \delta Z_2^K[\mathcal{L}^{(6)}] &= \frac{1}{F^4}\Big\{4(2m_K^4 - 2m_K^2m_\pi^2 + m_\pi^4)\beta_{17} \qquad (70)\\ &+ 4m_K^2(2m_K^2 + m_\pi^2)\beta_{18}\\ &+ 4(4m_K^4 - 4m_K^2m_\pi^2 + 3m_\pi^4)\beta_{19}\\ &+ 4(4m_K^4 + 4m_K^2m_\pi^2 + m_\pi^4)\beta_{21}\Big\}.\end{aligned}$$

If the unrenormalized contributions of the order p^2, p^4 and p^6 -diagrams of Fig. 2 are denoted as $\Delta_0 f$, $\Delta_1 f$, and $\Delta_2 f$, then, with the mass and wavefunction renormalizations given above, the renormalized form factors read

$$\begin{aligned}f &= \sqrt{Z^K}(\Delta_0 f + \Delta_1 f + \Delta_2 f)\sqrt{Z^\pi} \qquad (71)\\ &= \Delta_0 f + \left\{\Delta_1 f \right.\\ &\qquad \left. + \Delta_0 f\left(\frac{1}{2}\delta Z_1^K + \frac{1}{2}\delta Z_1^\pi\right)\right\}_{m^2 = m_{\text{ph}}^2 - \delta m^2,\, F = F_\pi - \delta F}\\ &+ \Delta_2 f + \Delta_1 f\left\{\frac{1}{2}\delta Z_1^K + \frac{1}{2}\delta Z_1^\pi\right\}\\ &+ \Delta_0 f\left\{\frac{1}{2}\delta Z_2^K + \frac{1}{2}\delta Z_2^\pi + \frac{1}{4}\delta Z_1^K\delta Z_1^\pi - \frac{1}{8}(\delta Z_1^K)^2\right.\\ &\left. - \frac{1}{8}(\delta Z_1^\pi)^2\right\} + \mathcal{O}(p^8),\end{aligned}$$

where f stands for $f_\pm$.

3.5 $\mathcal{L}^{(6)}$ contributions to the form factors

In every order of chiral perturbation theory there appear new operators with a priori unknown coefficients. The ten

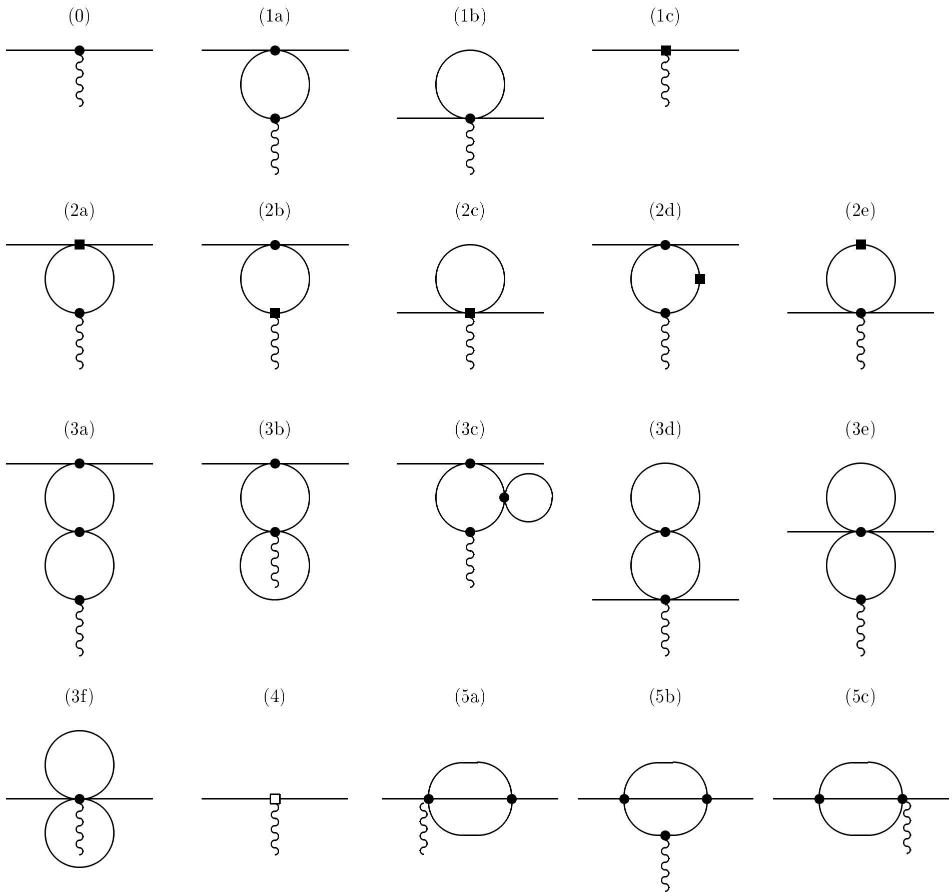

Fig. 2. The diagrams for the K_{l3} form factor up to order p^6. $\mathcal{L}^{(2)}$ vertices are denoted by filled circles (●), $\mathcal{L}^{(4)}$ vertices by filled squares (■), and an $\mathcal{L}^{(6)}$ vertex by an open square (□). Diagrams (0) to (3f) are referred to as "reducible", diagrams (5a) to (5c) as "irreducible"

constants of $\mathcal{L}^{(4)}$ are by now all fixed by experiment, but little is known about the 143 constants of $\mathcal{L}^{(6)}$. Out of the latter, only 11 enter in semileptonic K decay. There are two sources which lead to $\mathcal{L}^{(6)}$ contributions to the form factors: one is the $\mathcal{O}(p^6)$ tree graph of Fig. 2, and the other one is the $\mathcal{O}(p^2)$ tree graph of Fig. 2 with the $\mathcal{O}(p^6)$ wavefunction renormalization. Since to $\mathcal{O}(p^2)$

$$\begin{aligned} \Delta_0 f_+ &= 1, \\ \Delta_0 f_- &= 0, \end{aligned} \tag{72}$$

the total contribution involving $\mathcal{L}^{(6)}$ constants (ordered in powers of q^2) to the form factors f_+ and f_- becomes

$$\begin{aligned} \Delta f_+[\mathcal{L}^{(6)}] = {} & \frac{2q^4}{F^4}\{\beta_{22}+\beta_{23}\} - \frac{2q^2}{F^4}\Big\{\beta_{22}(m_K^2+m_\pi^2) \\ & - 2\beta_{24}m_K^2 - \beta_{25}m_\pi^2 - \beta_{26}(2m_K^2+m_\pi^2) \\ & + \beta_{27}(m_K^2+m_\pi^2)\Big\} + \frac{4}{F^4}\beta_{14}(m_K^2-m_\pi^2)^2, \end{aligned} \tag{73}$$

$$\begin{aligned} \Delta f_-[\mathcal{L}^{(6)}] = {} & -\frac{2q^2}{F^4}(m_K^2-m_\pi^2)\{\beta_8+\beta_{22}+\beta_{23}\} \\ & + \frac{m_K^2-m_\pi^2}{F^4}\Big\{2\beta_8(m_K^2+m_\pi^2) \\ & - 2\beta_{16}(2m_K^2+m_\pi^2) - 4\beta_{17}m_K^2 \\ & - 2\beta_{18}(2m_K^2+m_\pi^2) \\ & + 2\beta_{22}(m_K^2+m_\pi^2) - 4\beta_{24}m_K^2 \\ & - 2\beta_{25}m_\pi^2 - 2\beta_{26}(2m_K^2+m_\pi^2) \\ & + 2\beta_{27}(m_K^2+m_\pi^2)\Big\}. \end{aligned} \tag{74}$$

The q^4 term of $\Delta f_+[\mathcal{L}^{(6)}]$, i.e. $2q^4(\beta_{22}+\beta_{23})/F^4$, is the same as that for the electromagnetic form factor of the charged pion (and kaon). One can therefore use data on the second derivative of the pion electromagnetic form factor at the origin, together with the $\mathcal{O}(p^6)$ loop calculation, to determine the combination $\beta_{22}+\beta_{23}$. Details are given in Appendix D. We find for the ε^0 -part

$$(\beta_{22}+\beta_{23})^{(0),\mathrm{GL}} = 0.61\times 10^{-4}, \tag{75}$$

where we have used the GL scheme (46) and chosen the mass scale $\mu = 770\,\mathrm{MeV}$. To these counterterm contributions, we have to add those of the loop diagrams involving $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ vertices.

3.6 Divergent two-loop contributions to $f_\pm$

We start with an analysis of the divergent two-loop contributions of Fig. 2 to the form factors. The $\mathcal{O}(p^6)$ pole terms in ε have to cancel in the sum of all loops and the tree graphs with an $\mathcal{L}^{(6)}$ vertex:

$$\left(\Delta_{p^6}^{\mathrm{loop}} f_\pm + \Delta f_\pm[\mathcal{L}^{(6)}]\right)_{\mathrm{div}} = 0. \tag{76}$$

Since the $\mathcal{L}^{(6)}$ tree graph part $\Delta f_\pm[\mathcal{L}^{(6)}]$ is a polynomial in masses and momenta, it follows that the loop part $\Delta^{\text{loop}}_{p^6} f_\pm$ must also be polynomial in masses and momenta, i.e. it cannot contain any logarithms thereof. This condition offers a good check of the calculation. In fact, we find that in the sum of all $\mathcal{O}(p^6)$ loop diagrams any non-polynomial terms in masses and momenta cancel:

$$\begin{aligned}\Delta^{\text{loop}}_{p^6} f_+ &= \frac{1}{96(4\pi F)^4\varepsilon^2}\Big\{21m_K^4 - 42m_K^2m_\pi^2 \\ &\quad - 34m_K^2q^2 + 21m_\pi^4 - 28m_\pi^2q^2 - 3q^4\Big\} \\ &\quad + \frac{1}{576(4\pi)^2F^4\varepsilon}\Big\{768L_1^{(0)}(-m_K^4 + 2m_K^2m_\pi^2 + 3m_K^2q^2 \\ &\qquad -m_\pi^4 + 3m_\pi^2q^2 - q^4) \\ &\quad + 384L_2^{(0)}(-m_K^4 + 2m_K^2m_\pi^2 - 3m_K^2q^2 \\ &\qquad -m_\pi^4 - 3m_\pi^2q^2 + q^4) \\ &\quad + 64L_3^{(0)}(-m_K^4 + 2m_K^2m_\pi^2 + 45m_K^2q^2 \\ &\qquad -m_\pi^4 + 9m_\pi^2q^2 - 9q^4) \\ &\quad + 768L_4^{(0)}(3m_K^4 - 6m_K^2m_\pi^2 + m_K^2q^2 + 3m_\pi^4 + m_\pi^2q^2) \\ &\quad + 576L_5^{(0)}(-3m_K^4 + 6m_K^2m_\pi^2 + m_K^2q^2 - 3m_\pi^4 + m_\pi^2q^2) \\ &\quad + 288L_9^{(0)}q^2(7m_K^2 + 5m_\pi^2 + q^2) \\ &\quad - \frac{1}{(4\pi)^2}[70m_K^4 - 140m_K^2m_\pi^2 + 167m_K^2q^2 + 70m_\pi^4 \\ &\qquad +125m_\pi^2q^2 - 12q^4]\Big\}\end{aligned} \tag{77}$$

and

$$\begin{aligned}\Delta^{\text{loop}}_{p^6} f_- &= \frac{1}{48(4\pi F)^4\varepsilon^2} \\ &\quad \times \{29m_K^4 + 18m_K^2m_\pi^2 - 27m_K^2q^2 - 47m_\pi^4 + 27m_\pi^2q^2\} \\ &\quad + \frac{1}{576(4\pi)^2F^4\varepsilon} \\ &\quad \times \Big\{768L_1^{(0)}(-7m_K^4 + 3m_K^2q^2 + 7m_\pi^4 - 3m_\pi^2q^2) \\ &\quad + 384L_2^{(0)}(-19m_K^4 + 7m_K^2q^2 + 19m_\pi^4 - 7m_\pi^2q^2) \\ &\quad + 64L_3^{(0)}(-145m_K^4 + 48m_K^2m_\pi^2 + 41m_K^2q^2 + 97m_\pi^4 \\ &\qquad -41m_\pi^2q^2) \\ &\quad + 384L_4^{(0)}(-26m_K^4 + 9m_K^2m_\pi^2 + 17m_\pi^4) \\ &\quad + 64L_5^{(0)}(-m_K^4 + 2m_K^2m_\pi^2 + 27m_K^2q^2 - m_\pi^4 - 27m_\pi^2q^2) \\ &\quad + 6912L_6^{(0)}(2m_K^4 - m_K^2m_\pi^2 - m_\pi^4) \\ &\quad + 9216L_7^{(0)}(m_K^4 - 2m_K^2m_\pi^2 + m_\pi^4) \\ &\quad + 2304L_8^{(0)}(5m_K^4 - 4m_K^2m_\pi^2 - m_\pi^4) \\ &\quad + 288L_9^{(0)}(-7m_K^4 + 2m_K^2m_\pi^2 - m_K^2q^2 + 5m_\pi^4 + m_\pi^2q^2) \\ &\quad + \frac{1}{(4\pi)^2}[11m_K^4 - 204m_K^2m_\pi^2 - 51m_K^2q^2 + 193m_\pi^4 \\ &\qquad +51m_\pi^2q^2]\Big\}.\end{aligned} \tag{78}$$

This is *not* the case for the group of reducible resp. the group of irreducible diagrams alone, only in their sum. $L_i^{(0)}$ are the ε^0 coefficients of the $\mathcal{L}^{(4)}$ constants, cf. (41).

The divergent parts $\Delta f_\pm[\mathcal{L}^{(6)}]$, given explicitly in terms of the $\mathcal{L}^{(6)}$ constants β_j in (73) and (74), have to be the negative of these expressions so that the whole $\mathcal{O}(p^6)$ prediction is finite. Since the $\mathcal{L}^{(6)}$ constants β_j do not know anything about the masses m_π^2, m_K^2, they must appear in (73) and (74) in such a way that the mass dependence of the divergent parts $\Delta f_\pm[\mathcal{L}^{(6)}]$ is produced from the explicit masses in (73) and (74) alone. In other words: the $\mathcal{L}^{(6)}$ constants themselves cannot contribute any masses. In Appendix C we list the resulting divergent parts for the relevant $\mathcal{L}^{(6)}$ constants β_j. The fact that they are independent of the masses is a consistency check between our calculation and the Lagrangian $\mathcal{L}^{(6)}$ from [5]. We also agree with the double chiral logs of [10].

3.7 Reducible loop diagrams

The reducible diagrams of Fig. 2 can be expressed in terms of one-loop integrals. In case of two loops they are of the form

$$\mu^{8-2D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\mathrm{d}^Dl}{(2\pi)^D}\frac{V(k^2,kp_1,kp_2,kl,l^2,lp_1,lp_2)}{P} \tag{79}$$

and in case of one loop

$$\mu^{4-D}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{V(k^2,kp_1,kp_2)}{P}. \tag{80}$$

Here V is a polynomial of its arguments and P represents a product of propagator factors that depends on the topology of the diagram. A first simplification of these integrals is achieved by replacing certain factors in the numerator according to

$$\begin{aligned}k^2 &= P(k,m^2) + m^2, \\ kq &= \frac{1}{2}[P(k+q,m^2) + m^2 - k^2 - q^2], \\ l^2 &= P(l,m^2) + m^2, \\ lq &= \frac{1}{2}[P(l+q,m^2) + m^2 - l^2 - q^2],\end{aligned} \tag{81}$$

where $P(k,m^2) = k^2 - m^2 + \mathrm{i}0$. The remaining integrals can be expressed through the one-loop one-point function $A_0(m^2)$ of (59), the one-loop two-point function

$$\begin{aligned}&B_0(q^2,m_1^2,m_2^2) \\ &= \mu^{4-D}\int\frac{\mathrm{d}^Dk}{\mathrm{i}(2\pi)^D}\frac{1}{[(k+q)^2-m_1^2][k^2-m_2^2]},\end{aligned} \tag{82}$$

and tensors and mass derivatives thereof.

In a two-loop calculation these functions have to be considered up to order ε^1 (where $D = 4-2\varepsilon$). The results are given in Appendix A. The reducible contributions to f_+ are given explicitly for each diagram in Appendix B.

4 Irreducible two-loop diagrams

In the irreducible diagrams 5a, 5b, 5c of Fig. 2 the two-loop integrations are not independent of each other as they were in the reducible graphs. That is why genuine two-loop functions enter the stage which cannot be expressed by one-loop integrals only.

Inserting the Feynman rules yields integrands with a similar structure as in (79)

$$\frac{V(k^2,kp_1,kp_2,kl,l^2,lp_1,lp_2)}{P(k+p_1,m_0^2)P(k+p_2,m_1^2)P(k+l,m_2^2)P(l,m_3^2)}, \quad (83)$$

where V is a polynomial of degree equal to the number of vertices. After canceling factors via

$$\begin{aligned} kp_1 &= \frac{1}{2}[P(k+p_1,m_1^2)+m_0^2-k^2-p_1^2], \quad (84)\\ kp_2 &= \frac{1}{2}[P(k+p_2,m_1^2)+m_1^2-k^2-p_2^2],\\ kl &= \frac{1}{2}[P(k+l,m_2^2)+m_2^2-k^2-l^2],\\ l^2 &= P(l,m_3^2)+m_3^2, \end{aligned}$$

we are left with reducible integrals which can be calculated analytically, and with some genuine two-loop integrals of the *sunset* topology, i.e. the three-point functions

$$\begin{aligned} &T_{\alpha_1,\alpha_2,\beta}(q^2;p_1^2,p_2^2;m_0^2,m_1^2,m_2^2,m_3^2) \quad (85)\\ &= \mu^{8-2D}\int\frac{\mathrm{d}^Dk}{\mathrm{i}(2\pi)^D}\frac{\mathrm{d}^Dl}{\mathrm{i}(2\pi)^D}\\ &\times\frac{(lp_1)^{\alpha_1}(lp_2)^{\alpha_2}(k^2)^{\beta}}{P(k+p_1,m_0^2)P(k+p_2,m_1^2)P(k+l,m_2^2)P(l,m_3^2)}, \end{aligned}$$

and the two-point functions

$$\begin{aligned} S_{\alpha,\beta}(p^2;m_1^2,m_2^2,m_3^2) &= \mu^{8-2D}\int\frac{\mathrm{d}^Dk}{\mathrm{i}(2\pi)^D}\frac{\mathrm{d}^Dl}{\mathrm{i}(2\pi)^D} \quad (86)\\ &\times\frac{(lp)^{\alpha}(k^2)^{\beta}}{P(k+p,m_1^2)P(k+l,m_2^2)P(l,m_3^2)}. \end{aligned}$$

In diagram (5b) of Fig. 2, nine different mass flows of intermediate mesons must be regarded:

$$\begin{aligned} \Delta^{(5b)}f_\pm &= \Delta^{(5b)}_{K\pi\pi\pi}f_\pm + \Delta^{(5b)}_{K\pi KK}f_\pm + \Delta^{(5b)}_{K\pi\eta\eta}f_\pm \quad (87)\\ &+ \Delta^{(5b)}_{\pi K\pi K}f_\pm + \Delta^{(5b)}_{\pi KK\eta}f_\pm + \Delta^{(5b)}_{\eta K\pi K}f_\pm\\ &+ \Delta^{(5b)}_{\eta KK\eta}f_\pm + \Delta^{(5b)}_{K\eta KK}f_\pm + \Delta^{(5b)}_{K\eta\pi\eta}f_\pm, \quad (88) \end{aligned}$$

where $\Delta^{(5b)}_{rstu}f_\pm$ means that mesons r and s couple to the W boson and the other two lines are mesons of type t and u. Each mass flow is handled separately, and its contribution to the K_{l3} form factor is expressed in terms of the basic one- and two-loop functions A, B, $S_{\alpha,\beta}$, $T_{\alpha_1,\alpha_2,\beta}$, where for the latter at most the tensor indices

$S_{2,0}$, $S_{1,1}$, $S_{1,0}$, $S_{0,0}$, $S_{0,1}$, $S_{0,2}$, $T_{0,0,0}$, $T_{0,0,1}$, $T_{0,0,2}$, $T_{0,0,3}$, $T_{1,0,0}$, $T_{1,0,1}$, $T_{1,0,2}$, $T_{1,1,0}$, $T_{1,1,1}$

are needed. Except for special kinematic situations the genuine two-loop integrals $S_{\alpha,\beta}$ and $T_{\alpha_1,\alpha_2,\beta}$ cannot be calculated analytically. In [12] we describe the method by which we calculated them by splitting them up into one part which contains the divergence and can be evaluated analytically, and a second part which is finite and can be done numerically.

5 Finite contributions of the loops

After presenting the results of the pole terms of the loop diagrams we now come to their finite parts which contain the actual physical information. We will present the results, which can only be given in numerical form, graphically and as interpolation polynomials. We use the GL scheme discussed above at a scale $\mu = m_\rho = 770\,\mathrm{MeV}$, the masses given in (65), and the following values for the finite parts L_i^{ren} of the $\mathcal{L}^{(4)}$ constants:

i	1	2	3	4	5	6	7	8	9
$10^4 L_i^{\mathrm{ren}}$	7 ± 5	12 ± 4	-35 ± 13	-3 ± 5	14 ± 5	-2 ± 3	-4 ± 2	9 ± 3	69 ± 7

(89)

The ε^1 coefficients of L_i are not new degrees of freedom, but always appear in combination with certain $\mathcal{L}^{(6)}$ parameters. Therefore, we define

$$L_i^{(1),\mathrm{GL}}(\mu = m_\rho) = 0, \quad (90)$$

cf. (44). As our kinematical range we choose momentum transfers $-m_K^2 \le q^2 \le (m_K - m_\pi)^2$, which is certainly within the range of applicability of chiral perturbation theory. The results of the $\mathcal{O}(p^4)$ and the $\mathcal{O}(p^6)$ loop contributions (reducible and irreducible) for the form factors $f_+(q^2)$ and $f_-(q^2)$ are plotted as a function of $x = q^2/m_K^2$ in Figs. 3 and 4. We observe that for f_+ the reducible and the irreducible $\mathcal{O}(p^6)$ contributions cancel almost completely. For f_- the irreducible loop corrections are very small. For both f_+ and f_- the $\mathcal{O}(p^6)$ loop corrections are essentially linear functions of q^2, i.e. they create only small nonlinear contributions to the form factors.

The interpolation polynomials for the loop corrections at order p^6 of the form factors $f_\pm$ (i.e. the $\mathcal{O}(p^4,p^6)$ parts of the full renormalized form factors given in (71), but without the tree graphs from $\mathcal{L}^{(6)}$) read

$$\begin{aligned} \Delta^{\mathrm{loop}}_{p^6}f_+ &= x(-0.0101+0.0009x+0.0054x^2 \quad (91)\\ &+0.0007x^3)+\Delta^{\mathrm{loop}}_{p^6}f_+(0),\\ \Delta^{\mathrm{loop}}_{p^6}f_- &= x(0.0185+0.0001x+0.0022x^2 \quad (92)\\ &+0.0008x^3)+\Delta^{\mathrm{loop}}_{p^6}f_-(0), \end{aligned}$$

$x = q^2/m_K^2 \in [0, 0.5]$. For completeness, we quote the results for $f_\pm(0)$ which come from the $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ loops:

$$\begin{aligned} &\Delta^{\mathrm{loop}}_{p^4+p^6}f_+(0) \quad (93)\\ &= \Delta_{p^4}f_+(0) + \Delta^{\mathrm{red.loop}}_{p^6}f_+(0) + \Delta^{\mathrm{irred.loop}}_{p^6}f_+(0) \end{aligned}$$

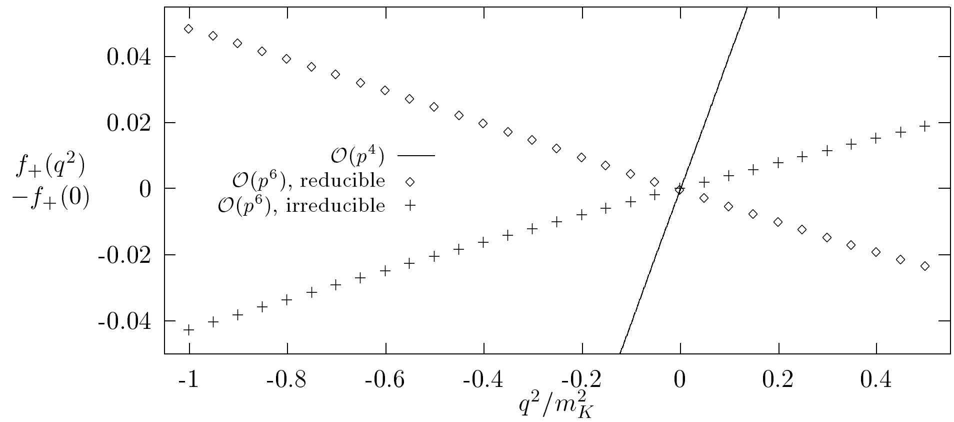

Fig. 3. Loop contributions to the form factor $f_+(q^2)$. The solid line is the leading order $\mathcal{O}(p^4)$ result, the dotted curves denote the corrections at order p^6 due to the reducible respectively irreducible loop diagrams in the GL scheme (on the one hand diagrams (2a)–(3f) and the p^6 contributions of the renormalized leading order diagrams (0), (1a)–(1c), on the other hand diagrams (5a)–(5c), cf. Fig. 2). The contribution of diagram (4) is missing: it would add an arbitrary parabola through the origin

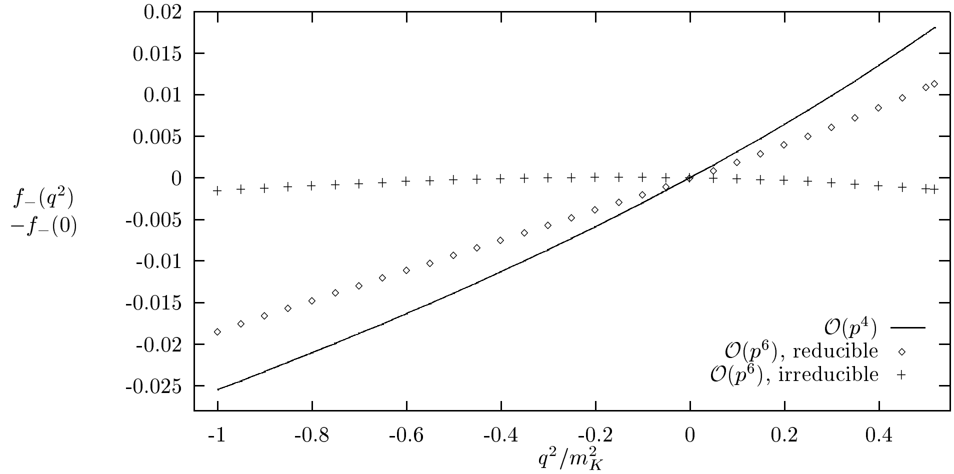

Fig. 4. Loop contributions to the form factor $f_-(q^2)$, in analogy to Fig. 3. The contribution from the $\mathcal{L}^{(6)}$ tree graph (4) is missing. It would add an arbitrary straight line through the origin

$$= -0.0229 + 0.00937 + 0.00853 = -0.0050,$$

$$\Delta^{\text{loop}}_{p^4+p^6} f_-(0) \tag{94}$$
$$= \Delta_{p^4} f_-(0) + \Delta^{\text{red.loop}}_{p^6} f_-(0) + \Delta^{\text{irred.loop}}_{p^6} f_-(0)$$
$$= -0.1836 + 0.0832 - 0.0533 = -0.1537.$$

Here, $\Delta^{\text{irred.loop}}_{p^6}$ denotes the contributions of the irreducible diagrams (5a), (5b), (5c) from Fig. 2 and the irreducible part of the wavefunction renormalization (diagram (h) from Fig. 1). $\Delta^{\text{red.loop}}_{p^6}$ stands for the remaining loop contributions which all come from reducible diagrams. To these results, the contributions of the $\mathcal{L}^{(6)}$ constants, which occur at $\mathcal{O}(p^6)$, must be added. These can either be determined by experiment or by model calculations.

The $\mathcal{O}(p^6)$ loop contributions still depend on the mass scale μ. This dependence follows from their divergent parts which are given in (77). For $f_+(0)$ we find a value of 0.0617 at the scale $\mu = 500$ MeV and a value of -0.0364 at the scale $\mu = 1000$ MeV to be compared with the above value, -0.0050, at $\mu = 770$ MeV. It is seen that the loop contribution to $f_\pm(0)$ is rather sensitive to the choice of μ due to cancellations of the reducible and the irreducible contributions. This μ dependence is of course canceled by the μ dependence of the p^6 counterterms. Equations (93) and (94) only serve as an indication of the size of the effects to be expected.

At the present stage the predictive power of our $\mathcal{O}(p^6)$ calculation lies only in quantifying a deviation from linear form factor rise. From (73) and (75) we find a nonlinear contribution to f_+ of

$$\Delta^{\text{nonlin.}}_{p^6} f_+(q^2) = 0.10 \frac{q^4}{m_K^4}, \tag{95}$$

whereas f_- contains negligible nonlinearities in the relevant kinematic range. The effect of the quadratic term in f_+ is essentially a lowering of the parameter λ_+ defined in (4). In our fit we find

$$\lambda_+ = 0.022 \tag{96}$$

(cf. Fig. 5) as compared to the linear fit $\lambda_+ = 0.0245$ in [2].

We conclude this section with a short discussion of the errors of the $\mathcal{O}(p^6)$ correction. Apart from the general ambiguity due to the $\mathcal{L}^{(6)}$ counterterms, errors arise from the $\mathcal{L}^{(4)}$ constants which appear in the $\mathcal{O}(p^6)$ corrections, and from the one $\mathcal{L}^{(6)}$ constant, associated with the q^4 term of $f_+(q^2)$, which can be extracted from the pion electromagnetic form factor. Although these errors could be as large as 10%, they are irrelevant to our result as the total $\mathcal{O}(p^6)$ effect is small; see Fig. 5.

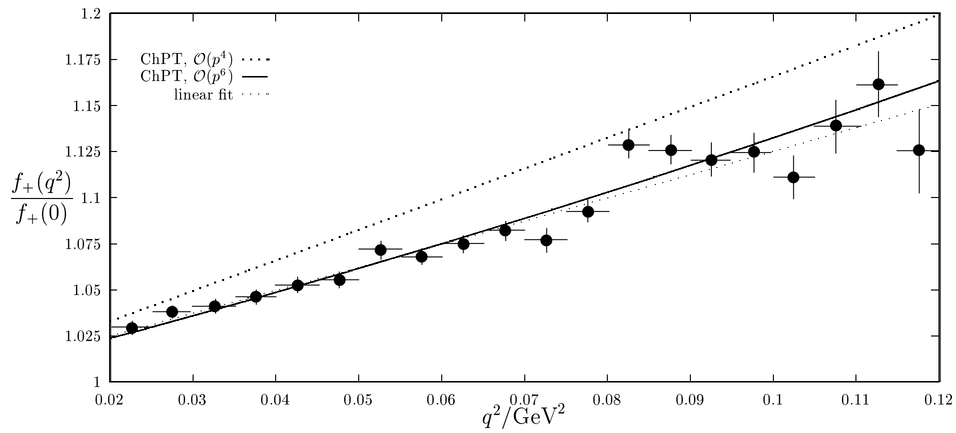

Fig. 5. The kaon semileptonic form factor f_+ to $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ chiral perturbation theory versus experiment. The data are from the CPLEAR experiment [2], the upper dotted curve is the leading order $\mathcal{O}(p^4)$ result of chiral perturbation theory, the solid curve is the full result at order p^6, and the lower dotted curve is the linear fit from [2]. The slope at the origin of the $\mathcal{O}(p^6)$ result has been fitted to the data; its second derivative comes from the electromagnetic pion form factor

6 Analysis of results and conclusions

We have calculated the $\mathcal{O}(p^6)$ contribution to semileptonic K_{l3} decay in $SU(3)\times SU(3)$ chiral perturbation theory. This is an effective field theory, so that there appear new operators with unknown couplings in each order of perturbation theory. For the K_{l3} form factors $f_\pm(q^2)$ this means that the constant and the linear term in q^2 are not determined by the theory. The q^4 counterterm, however, is the same for the semileptonic form factor $f_+(q^2)$ and the electromagnetic form factor of the charged pion. It can therefore be extracted by comparing the $\mathcal{O}(p^6)$ chiral perturbation theory with low-energy data on the pion electromagnetic form factor. The details are described in Appendix D. There is no q^4 counterterm for the form factor $f_-(q^2)$.

We have found that the $\mathcal{O}(p^6)$ loop corrections are essentially linear in q^2. Thus, the only nonlinearity is the q^4 contribution to f_+ coming from the $\mathcal{L}^{(6)}$ tree graph which is related to the electromagnetic pion form factor.

It is interesting to consider the reducible and the irreducible $\mathcal{O}(p^6)$ loop results separately as plotted in Figs. 3 and 4. It is clearly seen that for f_+ both terms cancel almost exactly in the physical range of K_{l3} decay. It should be kept in mind, however, that an arbitrary linear term can always be added to the $\mathcal{O}(p^6)$ predictions. For f_- the irreducible contributions are very small.

The results obtained are interesting from the following points of view.

(1) The convergence of chiral perturbation theory in semileptonic K_{l3} decay is established. It turns out that in K_{l3} decay the $\mathcal{O}(p^6)$ corrections are small. This need not always be the case [12].
(2) For the form factors, the deviation from a linear rise in q^2 is small, but not negligible (for f_+). The result of the nonlinear contribution to f_+ is effectively a lowering of the parameter λ_+.
(3) Our method of calculating the irreducible two-loop diagrams of high tensorial rank and involving three different masses can also be applied elsewhere.
(4) The wavefunction renormalization constants calculated here can be employed in other processes.
(5) The divergent parts of the relevant $\mathcal{L}^{(6)}$ parameters β_j appear in other processes as well and can be used there as a check.

Appendix

A One-loop integrals

In this appendix we reproduce the well-known one-loop results. With the definition

$$\mathrm{Lg}(m^2) = \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma \tag{97}$$

one has

$$\begin{aligned} A_0(m^2) &= \mu^{4-D}\int \frac{\mathrm{d}^D k}{\mathrm{i}(2\pi)^D}\frac{1}{k^2-m^2} \\ &= -\frac{\mu^2}{4\pi}\Gamma(-1+\varepsilon)\left(\frac{m^2}{4\pi\mu^2}\right)^{1-\varepsilon} \\ &= \frac{m^2}{(4\pi)^2}\left\{\frac{1}{\varepsilon}+1-\mathrm{Lg}(m^2)\right. \\ &\quad \left. +\ \varepsilon\left[1+\frac{\pi^2}{12}-\mathrm{Lg}(m^2)+\frac{\mathrm{Lg}(m^2)^2}{2}\right]+\mathcal{O}(\varepsilon^2)\right\}, \end{aligned} \tag{98}$$

and

$$\begin{aligned} &B_0(q^2;m_1^2,m_2^2) \\ &= \mu^{4-D}\int \frac{d^D k}{\mathrm{i}(2\pi)^D}\frac{1}{[(k+q)^2-m_1^2][k^2-m_2^2]} \\ &= \frac{b^{(-1)}(q^2;m_1^2,m_2^2)}{\varepsilon}+b^{(0)}(q^2;m_1^2,m_2^2) \\ &+ \varepsilon b^{(1)}(q^2;m_1^2,m_2^2)+\mathcal{O}(\varepsilon^2), \end{aligned} \tag{99}$$

with

$$b^{(-1)}(q^2;m_1^2,m_2^2) = \frac{1}{(4\pi)^2}, \tag{100}$$

$$b^{(0)}(q^2;m_1^2,m_2^2) = \frac{1}{(4\pi)^2}$$

$$\times \left\{ 2 - \mathrm{Lg}(m_1^2) + \sum_{j=1}^{2} x_j \log\left(1 - \frac{1}{x_j}\right) \right\},$$

$$b^{(1)}(q^2; m_1^2, m_2^2) = \frac{1}{(4\pi)^2}$$

$$\times \left\{ 4 + \frac{\pi^2}{12} - 2\mathrm{Lg}(m_1^2) + \frac{\mathrm{Lg}(m_1^2)^2}{2} \right.$$

$$+ \frac{1}{2} \sum_{j=1}^{2} x_j \log\left(1 - \frac{1}{x_j}\right)$$

$$\times [4 - 2\mathrm{Lg}(q^2 - \mathrm{i}0) - \log(1 - x_j) - \log(-x_j)]$$

$$- x_1 [\log(1 - x_1)\log(1 - x_2) - \log(-x_1)\log(-x_2)]$$

$$+ (x_1 - x_2)\left[\log(x_2 - x_1)\log\left(1 - \frac{1}{x_2}\right) - \mathrm{Li}_2\left(\frac{1 - x_2}{x_1 - x_2}\right) \right.$$

$$\left.\left. + \ \mathrm{Li}_2\left(\frac{-x_2}{x_1 - x_2}\right)\right]\right\},$$

where $D = 4 - 2\varepsilon$ is the dimension of space-time, $x_{1/2}$ are given by

$$x_{1,2} = \frac{1}{2q^2}\left\{ m_2^2 - m_1^2 + q^2 \pm \sqrt{\lambda(m_1^2, m_2^2, q^2)} \right\}, \quad (101)$$

λ being the Källén function $\lambda(a, b, c) = (a - b - c)^2 - 4bc$, and where we have expanded to order ε^1. All masses carry an infinitesimal negative imaginary part.

Tensor integrals can be reduced to scalar ones by decomposing them with respect to Lorentz covariants. The notation is

$A_r(m^2)$ = coefficient of the tensor integral with r momenta k in the numerator (r even) (102)

$B_{rs}(q^2; m_1^2, m_2^2)$ = coefficient of the tensor integral with r momenta k in the numerator, and s factors of $g_{\mu\nu}$ on the rhs, (103)

e.g.

$$\mu^{4-D} \int \frac{\mathrm{d}^D k}{\mathrm{i}(2\pi)^D} \frac{k^\mu k^\nu}{k^2 - m^2} = g^{\mu\nu} A_2 \quad (104)$$

and

$$\mu^{4-D} \int \frac{\mathrm{d}^D k}{\mathrm{i}(2\pi)^D} \frac{k^\mu}{[(k+q)^2 - m_1^2][k^2 - m_2^2]} \quad (105)$$

$$= q^\mu B_{10},$$

$$\mu^{4-D} \int \frac{\mathrm{d}^D k}{\mathrm{i}(2\pi)^D} \frac{k^\mu k^\nu}{[(k+q)^2 - m_1^2][k^2 - m_2^2]}$$

$$= q^\mu q^\nu B_{20} + g^{\mu\nu} B_{21},$$

$$\mu^{4-D} \int \frac{\mathrm{d}^D k}{\mathrm{i}(2\pi)^D} \frac{k^\mu k^\nu k^\rho}{[(k+q)^2 - m_1^2][k^2 - m_2^2]}$$

$$= q^\mu q^\nu q^\rho B_{30} + (g^{\mu\nu} q^\rho + g^{\mu\rho} q^\nu + g^{\nu\rho} q^\mu) B_{31}.$$

A_2 is given in terms of A_0 in (64), and the functions B_{rs} can all be related to B_0 (and A_0). For the form factor f_+ we need

$$B_{21}(q^2; m_1^2, m_2^2) = \frac{1}{4q^2(1 - D)} \quad (106)$$

$$\times \left\{ (m_2^2 - m_1^2 - q^2) A_0(m_1^2) + (m_1^2 - m_2^2 - q^2) A_0(m_2^2) \right.$$

$$\left. + \lambda(q^2, m_1^2, m_2^2) B_0(q^2; m_1^2, m_2^2) \right\},$$

$$B_{31}(q^2; m_1^2, m_2^2) = \frac{1}{8q^4(1 - D)} \quad (107)$$

$$\times \left\{ \left[q^4 - m_1^4 - m_2^4 + 2m_1^2 m_2^2 + 4m_1^2 q^2 - \frac{4m_1^2 q^2}{D} \right] \right.$$

$$\times A_0(m_1^2) + \left[\lambda(q^2, m_1^2, m_2^2) + \frac{4m_2^2 q^2}{D} \right] A_0(m_2^2)$$

$$\left. + (m_1^2 - m_2^2 - q^2) \lambda(q^2, m_1^2, m_2^2) B_0(q^2; m_1^2, m_2^2) \right\}.$$

In addition, there are one-loop integrals which involve higher powers of propagators. These are obtained from the formulae above by differentiation with respect to m^2:

$$A_{0;n}(m^2) = \frac{1}{(n-1)!} \left[\frac{\mathrm{d}}{\mathrm{d}m^2} \right]^{n-1} A_0(m^2), \quad (108)$$

$$B_{rs;n_1 n_2}(q^2; m_1^2, m_2^2) = \frac{1}{(n_1 - 1)!(n_2 - 1)!} \quad (109)$$

$$\times \left[\frac{\partial}{\partial m_1^2} \right]^{n_1 - 1} \left[\frac{\partial}{\partial m_2^2} \right]^{n_2 - 1} B_{rs}(q^2; m_1^2, m_2^2).$$

B Reducible contributions to the $K_{\ell 3}$ form factor f_+

In this appendix we give the reducible contribution Δ of each diagram for the $K_{\ell 3}$ form factor f_+. The upper index of Δ refers to a specific diagram in Fig. 2.

The basic one-loop functions occurring here (and in the other form factor f_-) are A_0 and B_0, and tensors and mass derivatives thereof, cf. Appendix A.

$$\Delta^{(0)} f_+ = 1, \quad (110)$$

$$\Delta^{(1a)} f_+ = -\frac{3}{2F^2} \left\{ B_{21}(q^2; m_\eta^2, m_K^2) \right. \quad (111)$$

$$\left. + B_{21}(q^2; m_K^2, m_\pi^2) \right\},$$

$$\Delta^{(1b)} f_+ = \frac{1}{6F^2} \left\{ 3A_0(m_\eta^2) + 7A_0(m_K^2) \right. \quad (112)$$

$$\left. + 5A_0(m_\pi^2) \right\},$$

$$\Delta^{(1c)} f_+ = \frac{2}{F^2} \left\{ 4L_4(m_\pi^2 + 2m_K^2) \right. \quad (113)$$

$$\left. + 2L_5(m_\pi^2 + m_K^2) + q^2 L_9 \right\},$$

$$\Delta^{(2a)} f_+ = \frac{2}{F^4} \left\{ 2B_{31}(q^2; m_\eta^2, m_K^2) L_3(m_\pi^2 - m_K^2) \right. \quad (114)$$

$$- 2B_{31}(q^2; m_K^2, m_\pi^2)[8L_1 + 4L_2 + L_3](m_\pi^2 - m_K^2)$$

$$+ B_{21}(q^2;m_\eta^2,m_K^2)\big[L_3(m_\pi^2-m_K^2+3q^2) - 6L_4(m_\pi^2+2m_K^2) - 2L_5(m_\pi^2+5m_K^2)\big] - B_{21}(q^2;m_K^2,m_\pi^2) \times \big[8L_1(m_\pi^2-m_K^2-q^2) + 4L_2(m_\pi^2-m_K^2+q^2) + L_3(m_\pi^2-m_K^2-3q^2) + 2L_4(7m_\pi^2+10m_K^2) + 6L_5(m_\pi^2+m_K^2)\big]\Big\},$$

$$\Delta^{(2b)}f_+ = \frac{1}{F^4}\Big\{3B_{21}(q^2;m_K^2,m_\pi^2) \times \big[-4L_4(m_\pi^2+2m_K^2) - 2L_5(m_\pi^2+m_K^2) - q^2L_9\big] + B_{21}(q^2;m_\eta^2,m_K^2)\big[-12L_4(m_\pi^2+2m_K^2) + 2L_5(m_\pi^2-7m_K^2) - 3q^2L_9\big]\Big\}, \quad (115)$$

$$\Delta^{(2c)}f_+ = \frac{1}{3F^4}\Big\{A_0(m_\eta^2)[-48L_1m_\eta^2 - 14L_3m_\eta^2 + 4L_4(14m_K^2+m_\pi^2) + 13L_5(2m_K^2+m_\pi^2) + 3q^2L_9] + A_0(m_K^2)[-192L_1m_K^2 - 24L_2m_K^2 - 60L_3m_K^2 + 4L_4(44m_K^2+7m_\pi^2) + 2L_5(31m_K^2+7m_\pi^2) + 7q^2L_9] + A_0(m_\pi^2)[-144L_1m_\pi^2 - 24L_2m_\pi^2 - 54L_3m_\pi^2 + 4L_4(10m_K^2+29m_\pi^2) + 2L_5(2m_K^2+23m_\pi^2) + 5q^2L_9] - 2A_2(m_\eta^2)[24L_2+5L_3] - 12A_2(m_K^2)[4L_1+18L_2+5L_3] - 6A_2(m_\pi^2)[8L_1+28L_2+7L_3]\Big\}, \quad (116)$$

$$\Delta^{(2d)}f_+ = \frac{1}{F^4}\Big\{12B_{21}(q^2;m_\eta^2,m_K^2) \times [L_4(2m_K^2+m_\pi^2) + L_5m_K^2] + 4B_{21}(q^2;m_K^2,m_\eta^2) \times [3L_4(2m_K^2+m_\pi^2) + L_5(4m_K^2-m_\pi^2)] + 12B_{21}(q^2;m_K^2,m_\pi^2)[L_4(2m_K^2+m_\pi^2) + L_5m_\pi^2] + 12B_{21}(q^2;m_\pi^2,m_K^2)[L_4(2m_K^2+m_\pi^2) + L_5m_K^2] + 12B_{21;12}(q^2;m_\eta^2;m_K^2)m_K^2[L_4(2m_K^2+m_\pi^2) + L_5m_K^2 - 2L_6(2m_K^2+m_\pi^2) - 2L_8m_K^2] + 12B_{21;12}(q^2;m_K^2;m_\pi^2)m_\pi^2[L_4(2m_K^2+m_\pi^2) + L_5m_\pi^2 - 2L_6(2m_K^2+m_\pi^2) - 2L_8m_\pi^2] + 12B_{21;12}(q^2;m_\pi^2;m_K^2)m_K^2[L_4(2m_K^2+m_\pi^2) + L_5m_K^2 - 2L_6(2m_K^2+m_\pi^2) - 2L_8m_K^2] + 4B_{21;12}(q^2;m_K^2;m_\eta^2) \times [3L_4m_\eta^2(2m_K^2+m_\pi^2) + L_5m_\eta^2(4m_K^2-m_\pi^2) - 2L_6(8m_K^4+2m_K^2m_\pi^2-m_\pi^4) - 16L_7(m_K^2-m_\pi^2)^2 - 2L_8(8m_K^4-8m_K^2m_\pi^2+3m_\pi^4)]\Big\}, \quad (117)$$

$$\Delta^{(2e)}f_+ = \frac{1}{3F^4}\Big\{-4A_0(m_\eta^2) \times [3L_4(2m_K^2+m_\pi^2) + L_5(4m_K^2-m_\pi^2)] - 28A_0(m_K^2) \times [L_4(2m_K^2+m_\pi^2) + L_5m_K^2] - 20A_0(m_\pi^2) \times [L_4(2m_K^2+m_\pi^2) + L_5m_\pi^2] - 28A_{0;2}(m_K^2)m_K^2 \times [L_4(2m_K^2+m_\pi^2) + L_5m_K^2 - 2L_6(2m_K^2+m_\pi^2) - 2L_8m_K^2] - 20A_{0;2}(m_\pi^2)m_\pi^2 \times [L_4(2m_K^2+m_\pi^2) + L_5m_\pi^2 - 2L_6(2m_K^2+m_\pi^2) - 2L_8m_\pi^2] - 4A_{0;2}(m_\eta^2) \times [3L_4m_\eta^2(2m_K^2+m_\pi^2) + L_5m_\eta^2(4m_K^2-m_\pi^2) - 2L_6(8m_K^4+2m_K^2m_\pi^2-m_\pi^4) - 16L_7(m_K^2-m_\pi^2)^2 - 2L_8(8m_K^4-8m_K^2m_\pi^2+3m_\pi^4)]\Big\}, \quad (118)$$

$$\Delta^{(3a)}f_+ = \frac{9}{4F^4}\Big\{B_{21}(q^2;m_\eta^2,m_K^2)^2 + 2B_{21}(q^2;m_\eta^2,m_K^2)B_{21}(q^2;m_K^2,m_\pi^2) + B_{21}(q^2;m_K^2,m_\pi^2)^2\Big\}, \quad (119)$$

$$\Delta^{(3b)}f_+ = -\frac{1}{4F^4}\Big\{3A_0(m_\eta^2)B_{21}(q^2;m_\eta^2,m_K^2) + 3A_0(m_\eta^2)B_{21}(q^2;m_K^2,m_\pi^2) + 3A_0(m_\eta^2)B_{21}(q^2;m_K^2,m_\pi^2) + 9A_0(m_K^2)B_{21}(q^2;m_\eta^2,m_K^2) + 7A_0(m_K^2)B_{21}(q^2;m_K^2,m_\pi^2) + 3A_0(m_\pi^2)B_{21}(q^2;m_\eta^2,m_K^2) + 5A_0(m_\pi^2)B_{21}(q^2;m_K^2,m_\pi^2)\Big\}, \quad (120)$$

$$\Delta^{(3c)}f_+ = \frac{1}{24F^4}\Big\{9B_{21}(q^2;m_\eta^2,m_K^2) \times [A_0(m_\pi^2) + 6A_0(m_K^2) + A_0(m_\eta^2)] + B_{21}(q^2;m_\pi^2,m_K^2) \times [33A_0(m_\pi^2) + 30A_0(m_K^2) + 9A_0(m_\eta^2)] - 6B_{21;12}(q^2;m_K^2;m_\pi^2)m_\pi^2 \times [A_0(m_\eta^2) - 3A_0(m_\pi^2)] + 3B_{21;12}(q^2;m_\pi^2;m_K^2)A_0(m_\pi^2)(3m_\eta^2+m_\pi^2) + 3B_{21;12}(q^2;m_\eta^2;m_K^2)A_0(m_\eta^2)(3m_\eta^2+m_\pi^2) - 2B_{21;12}(q^2;m_K^2;m_\eta^2) \times [9m_\pi^2A_0(m_\pi^2) - 6(3m_\eta^2+m_\pi^2)A_0(m_K^2) + (16m_K^2-7m_\pi^2)A_0(m_\eta^2)\Big\}, \quad (121)$$

$$\Delta^{(3d)}f_+ = -\frac{1}{72F^4}\Big\{57A_0(m_\eta^2)A_0(m_K^2) + 42A_0(m_K^2)^2 + 41A_0(m_K^2)A_0(m_\pi^2) + 40A_0(m_\pi^2)^2 - 2A_{0;2}(m_\eta^2)A_0(m_\eta^2)(16m_K^2-7m_\pi^2) + 12A_{0;2}(m_\eta^2)A_0(m_K^2)(3m_\eta^2+m_\pi^2) - 18A_{0;2}(m_\eta^2)A_0(m_\pi^2)m_\pi^2 + 7A_{0;2}(m_K^2)A_0(m_\eta^2)(3m_\eta^2+m_\pi^2) - 10A_{0;2}(m_\pi^2)A_0(m_\eta^2)m_\pi^2 + 30A_{0;2}(m_\pi^2)A_0(m_\pi^2)m_\pi^2\Big\}, \quad (122)$$

$$\Delta^{(3e)}f_+ = -\frac{1}{18F^4}\Big\{9A_0(m_\eta^2)B_{21}(q^2;m_\eta^2,m_K^2) + 6A_0(m_\eta^2)B_{21}(q^2;m_K^2,m_\pi^2) \quad (123)$$

$$
\begin{aligned}
&+ 24A_0(m_K^2)B_{21}(q^2;m_\eta^2,m_K^2)\\
&+ 19A_0(m_K^2)B_{21}(q^2;m_K^2,m_\pi^2)\\
&+ 12A_0(m_\pi^2)B_{21}(q^2;m_\eta^2,m_K^2)\\
&+ 20A_0(m_\pi^2)B_{21}(q^2;m_K^2,m_\pi^2)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{(3f)}f_+ = \frac{1}{360F^4}\Big\{&81A_0(m_\eta^2)^2 \\
&+ 168A_0(m_\eta^2)A_0(m_K^2) + 132A_0(m_\eta^2)A_0(m_\pi^2)\\
&+ 318A_0(m_K^2)^2 + 386A_0(m_K^2)A_0(m_\pi^2) + 175A_0(m_\pi^2)^2\Big\}.
\end{aligned} \tag{124}
$$

C Divergent parts of $\mathcal{L}^{(6)}$ constants

In this appendix we list the divergent parts of all $\mathcal{L}^{(6)}$-constants which occur in the meson vector form factors. They are derived from (73) and (74) on the one hand and (77) and (78) on the other hand taking into consideration (76). Similar relations for the electromagnetic form factors of $\pi^\pm$, $K^\pm$, K_0, and the weak form factors $f_\pm$ for the $\eta \to K^\pm W^\mp$ decay are also taken into consideration [12, 16].

Let

$$
\beta_j = \frac{\beta_j^{(-2)}}{\varepsilon^2} + \frac{\beta_j^{(-1)}}{\varepsilon} + \beta_j^{(0)} + \mathcal{O}(\varepsilon) \tag{125}
$$

be the Laurent expansion of the $\mathcal{L}^{(6)}$ parameters, cf. (46). For their divergent parts

$$
\beta_j^{\mathrm{div}} = \frac{\beta_j^{(-2)}}{\varepsilon^2} + \frac{\beta_j^{(-1)}}{\varepsilon}, \tag{126}
$$

we find

$$
\begin{aligned}
\beta_{22}^{\mathrm{div}} + \beta_{23}^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{1}{64(4\pi)^2\varepsilon^2}\\
&- \frac{1}{\varepsilon}\left[\frac{1}{96(4\pi)^2} - \frac{2}{3}L_1^{(0)} + \frac{1}{3}L_2^{(0)} - \frac{1}{2}L_3^{(0)} + \frac{1}{4}L_9^{(0)}\right]\Bigg\},
\end{aligned} \tag{127}
$$

$$
2\beta_{24}^{\mathrm{div}} - \beta_{25}^{\mathrm{div}} = -\frac{1}{(4\pi)^2}\frac{3}{2\varepsilon}L_3^{(0)}, \tag{128}
$$

$$
\begin{aligned}
\beta_{26}^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{1}{32(4\pi)^2\varepsilon^2} + \frac{1}{\varepsilon}\\
&\times\left[\frac{7}{192(4\pi)^2} - \frac{1}{2}L_3^{(0)} - \frac{1}{2}L_9^{(0)}\right]\Bigg\},
\end{aligned} \tag{129}
$$

$$
\begin{aligned}
\beta_{22}^{\mathrm{div}} - 2\beta_{24}^{\mathrm{div}} + \beta_{27}^{\mathrm{div}} = &\frac{1}{(4\pi)^2}\\
&\times\Bigg\{\frac{-11}{96(4\pi)^2\varepsilon^2} - \frac{1}{\varepsilon}\left[\frac{83}{1152(4\pi)^2} - 2L_1^{(0)} + L_2^{(0)}\right.\\
&\left.-\frac{3}{2}L_3^{(0)} - \frac{2}{3}L_4^{(0)} - \frac{1}{2}L_5^{(0)} - \frac{3}{4}L_9^{(0)}\right]\Bigg\},
\end{aligned} \tag{130}
$$

$$
\begin{aligned}
\beta_{14}^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{-7}{128(4\pi)^2\varepsilon^2} + \frac{1}{\varepsilon}\left[\frac{35}{1152(4\pi)^2}\right.\\
&\left.+ \frac{1}{3}L_1^{(0)} + \frac{1}{6}L_2^{(0)} + \frac{1}{36}L_3^{(0)} - L_4^{(0)} + \frac{3}{4}L_5^{(0)}\right]\Bigg\},
\end{aligned} \tag{131}
$$

$$
\begin{aligned}
\beta_8^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{-19}{64(4\pi)^2\varepsilon^2} - \frac{1}{\varepsilon}\left[\frac{13}{384(4\pi)^2}\right.\\
&\left.- \frac{4}{3}L_1^{(0)} - \frac{8}{3}L_2^{(0)} - \frac{16}{9}L_3^{(0)} - \frac{3}{2}L_5^{(0)}\right]\Bigg\},
\end{aligned} \tag{132}
$$

$$
\begin{aligned}
\beta_{16}^{\mathrm{div}} + \beta_{18}^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{3}{64(4\pi)^2\varepsilon^2}\\
&- \frac{1}{\varepsilon}\left[\frac{119}{384(4\pi)^2} + \frac{4}{3}L_1^{(0)} + \frac{14}{3}L_2^{(0)} + \frac{28}{9}L_3^{(0)} + 5L_4^{(0)}\right.\\
&\left.- \frac{37}{18}L_5^{(0)} - 6L_6^{(0)} + 8L_7^{(0)} - 2L_8^{(0)}\right]\Bigg\},
\end{aligned} \tag{133}
$$

$$
\begin{aligned}
\beta_{17}^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{-17}{128(4\pi)^2\varepsilon^2}\\
&+ \frac{1}{\varepsilon}\left[\frac{173}{768(4\pi)^2} + \frac{2}{3}L_1^{(0)} + \frac{7}{3}L_2^{(0)} + \frac{11}{9}L_3^{(0)} + L_4^{(0)}\right.\\
&\left.- \frac{13}{12}L_5^{(0)} + 12L_7^{(0)} + 3L_8^{(0)}\right]\Bigg\},
\end{aligned} \tag{134}
$$

$$
\begin{aligned}
\beta_{15}^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{7}{192(4\pi)^2\varepsilon^2} - \frac{1}{\varepsilon}\left[\frac{5}{324(4\pi)^2}\right.\\
&\left.+ \frac{2}{9}L_1^{(0)} + \frac{1}{9}L_2^{(0)} + \frac{1}{54}L_3^{(0)} - \frac{2}{3}L_4^{(0)} + \frac{1}{2}L_5^{(0)}\right]\Bigg\},
\end{aligned} \tag{135}
$$

$$
\begin{aligned}
\beta_{20}^{\mathrm{div}} = \frac{1}{(4\pi)^2}\Bigg\{&\frac{-11}{384(4\pi)^2\varepsilon^2}\\
&- \frac{1}{\varepsilon}\left[\frac{125}{2304(4\pi)^2} + \frac{2}{9}L_1^{(0)} + \frac{7}{9}L_2^{(0)} + \frac{17}{108}L_3^{(0)} + \frac{1}{3}L_4^{(0)}\right.\\
&\left.- \frac{17}{12}L_5^{(0)} + 4L_7^{(0)} + 2L_8^{(0)}\right]\Bigg\},
\end{aligned} \tag{136}
$$

where $L_i^{(0)}$ are the ε^0 coefficients of the $\mathcal{L}^{(4)}$ parameters, cf. (41).

Note that the β_j do not contain any mass terms. Thus, the only source of masses in the Lagrangian (19) is the mass matrix χ defined in (10).

D Electromagnetic pion form factor

The $\mathcal{O}(p^6)$ calculation of the pion electromagnetic form factor is carried out in a manner completely analogous to that of the kaon semi-leptonic form factors. We quote the result of the $\mathcal{L}^{(6)}$ contribution:

$$
\begin{aligned}
F^{\pi^+}[\mathcal{L}^{(6)}] = &\frac{2q^4}{F^4}\{\beta_{22} + \beta_{23}\}\\
&+ \frac{2q^2}{F^4}\{-2m_\pi^2\beta_{22} + 2m_\pi^2\beta_{24}\\
&+ m_\pi^2\beta_{25} + (m_\pi^2 + 2m_K^2)\beta_{26} - 2m_\pi^2\beta_{27}\}.
\end{aligned} \tag{137}
$$

For the interpolation polynomials of the $\mathcal{O}(p^6)$ loop corrections (reducible plus irreducible diagrams) we obtain

$$
\begin{aligned}
\Delta_{p^6}^{\mathrm{loop}}[F^{\pi^+}] = x(&0.0813 + 0.0619x + 0.0387x^2\\
&+ 0.0218x^3 + 0.0068x^4)
\end{aligned} \tag{138}
$$

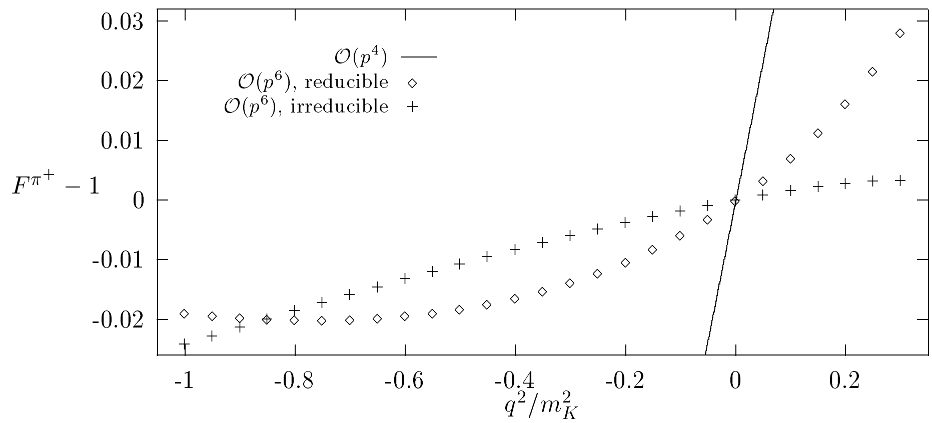

Fig. 6. $\mathcal{O}(p^6)$ loop contributions to the pion electromagnetic form factor, in analogy to Fig. 3

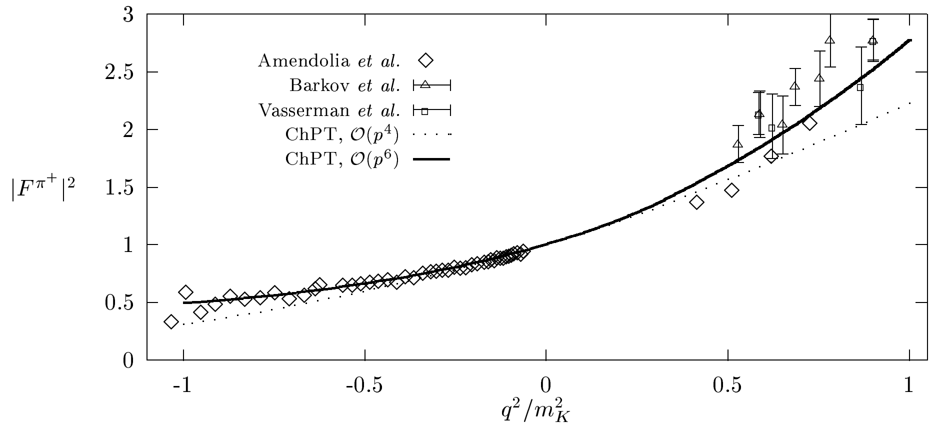

Fig. 7. The pion electromagnetic form factor in $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$ chiral perturbation theory versus experiment. The data are from [21–23]

for $x = q^2/m_K^2 \in [-1, (2m_\pi)^2/m_K^2])$ and in the decay region

$$\begin{aligned}\Delta_{p^6}^{\text{loop}}[F^{\pi^+}] = &-0.0084 + 0.1076x + 0.0956x^2 \\ &- 0.0544x^3 + 0.0233x^4 - 0.0047x^5 \\ &+ \mathrm{i}(0.0120 - 0.0998x + 0.2435x^2 + 0.1848x^3 \\ &+ 0.1087x^4 + 0.0260x^5)\end{aligned} \tag{139}$$

for $x = q^2/m_K^2 \in [4m_\pi^2/m_K^2, 1])$.

The separate reducible and irreducible $\mathcal{O}(p^6)$ loop contributions (modulo a quadratic polynomial) to the pion form factor are plotted in Fig. 6.

The arbitrary linear term can be fitted by using the experimental pion charge radius [21]

$$\left\langle r^2 \right\rangle^{\pi^+} = (0.439 \pm 0.008)\,\mathrm{fm}^2. \tag{140}$$

The constant multiplying q^4 is obtained by using the curvature of the form factor at the origin. We chose a value of $3.4\,\mathrm{GeV}^{-4}$, which represents an average of some typical extractions from experiment [18–20]. As Fig. 7 demonstrates this choice of parameters yields a good description of the experimental form factor.

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